Self Evaluation Test

1. A is $n \times n$ normal matrix. If λ is characteristic value of A corresponding to characteristic vector v then $\overline{\lambda}$ is the characteristic value of A^* with the same characteristic vector v.

Solution. Let B is any normal matrix. Then

$$\begin{aligned}\|Bv\|^2 &= \langle Bv, Bv\rangle \\ &= \langle v, B^*Bv\rangle \\ &= \langle v, BB^*v\rangle \quad [\because B \text{ is normal}] \\ &= \langle B^*v, B^*v\rangle \\ &= \|B^*v\|^2\end{aligned}$$

Take $B = A - \lambda I$. Note that B is normal as A is normal. Thus

$$\begin{aligned}\|(A-\lambda I)v\| &= \|(A-\lambda I)^*v\| \\ &= \|(A^* - \overline{\lambda} I)v\|\end{aligned}$$

So that $(A - \lambda I)v = 0$ if and only if $(A^* - \overline{\lambda} I)v = 0$.

2. Let $X = \mathbb{R}^3$ and define $f_1, f_2, f_3 \in X^*$ as follows:

$$f_1(x, y, z) = x - 2y$$

$$f_2(x, y, z) = x + y + z$$

$$f_3(x, y, z) = y - 3z$$

Show that f_1, f_2, f_3 is a basis of X^* and then find a basis for X for which it is the dual basis.

Solution. Firstly we will show that $\{f_1, f_2, f_3\}$ forms a basis for X^*. i.e.

(i) f_i are linear functional on X.

$f_i : X \to F$ (by definition)

and linearity can be checked easily. dim(X) =dim$(X^*) = 3$.

(ii) Linearly independent.

Construct $A_{3\times 3}$ by taking coefficients of f_i in row,

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -3 \end{bmatrix} \text{ and } c_1 f_1 + c_2 f_2 + c_3 f_3 = 0 \;\Leftrightarrow\; A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0.$$

Since $\det(A) \neq 0$ this homogenous system has only trivial solution. So f_1, f_2, f_3 are linearly independent. Hence $\{f_1, f_2, f_3\}$ is the basis for X^*.

Now let $\{f_1, f_2, f_3\}$ is dual basis of $\{\overline{x_1}, \overline{x_2}, \overline{x_3}\}$ where $\overline{x_1} = (x_i, y_i, z_i) \in X$.

$$\Rightarrow \; f_i(\overline{x_j}) = \delta_{ij}$$

So we get the system

$$\begin{aligned} A[\overline{x_1}, \overline{x_2}, \overline{x_3}] &= \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -3 \end{bmatrix} [\overline{x_1}, \overline{x_2}, \overline{x_3}] \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \text{So } [\overline{x_1}, \overline{x_2}, \overline{x_3}] &= A^{-1} I \\ &= A^{-1} \\ \overline{x_1}, \overline{x_2}, \overline{x_3} &= \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{-3}{10} & \frac{3}{10} & \frac{1}{10} \\ \frac{-1}{10} & \frac{1}{10} & \frac{-3}{10} \end{bmatrix} \end{aligned}$$

$$\overline{x_1} = \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \\ \frac{1}{5} \end{bmatrix}, \; \overline{x_2} = \begin{bmatrix} \frac{-3}{10} \\ \frac{3}{10} \\ \frac{1}{10} \end{bmatrix}, \; \overline{x_3} = \begin{bmatrix} \frac{-1}{10} \\ \frac{1}{10} \\ \frac{-3}{10} \end{bmatrix}$$

3. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ is normal i.e. $AA^* = A^A$. Then show that there exist a Unitary matrix U such that $U^*AU = D$ where D is diagonal matrix.

Solution. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ with $\chi_A(\theta) = \theta^3 - 3\theta^2 + 3\theta - 2$ and eigen values are 2, $\frac{1 \pm \iota\sqrt{3}}{2}$ i.e. $\lambda_1 = 2$, $\lambda_2 = \frac{1 + \iota\sqrt{3}}{2}$, $\lambda_3 = \frac{1 - \iota\sqrt{3}}{2}$ for which we have eigen vectors

$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 1 \\ \dfrac{\iota\sqrt{3}-1}{2} \\ -\left(\dfrac{1+\iota\sqrt{3}}{2}\right) \end{pmatrix}$, $x_3 = \begin{pmatrix} 1 \\ -\left(\dfrac{1+\iota\sqrt{3}}{2}\right) \\ \dfrac{\iota\sqrt{3}-1}{2} \end{pmatrix}$ which are orthogonal with $\|x_i\| = \sqrt{3}$ and $\dfrac{1}{\sqrt{3}}x_i$ are orthonormal.

So take $U = \dfrac{1}{\sqrt{3}}[x_1\ x_2\ x_3]$ which is unitary and $U^*AU = D$ i.e.

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \dfrac{-1-\iota\sqrt{3}}{2} & -\left(\dfrac{1-\iota\sqrt{3}}{2}\right) \\ 1 & -\left(\dfrac{1-\iota\sqrt{3}}{2}\right) & \dfrac{-1-\iota\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \dfrac{\iota\sqrt{3}-1}{2} & -\left(\dfrac{+\iota\sqrt{3}}{2}\right) \\ 1 & -\left(\dfrac{1+\iota\sqrt{3}}{2}\right) & \dfrac{\iota\sqrt{3}-1}{2} \end{bmatrix} =$$

$$\frac{1}{3}\begin{bmatrix} 2 & 2 & 2 \\ \dfrac{1+\iota\sqrt{3}}{2} & \dfrac{1-\iota\sqrt{3}}{2} & -1 \\ \dfrac{1-\iota\sqrt{3}}{2} & \dfrac{1+\iota\sqrt{3}}{2} & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & \dfrac{\iota\sqrt{3}-1}{2} & -\left(\dfrac{+\iota\sqrt{3}}{2}\right) \\ 1 & -\left(\dfrac{1+\iota\sqrt{3}}{2}\right) & \dfrac{\iota\sqrt{3}-1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \dfrac{1+\iota\sqrt{3}}{2} & 0 \\ 0 & 0 & \dfrac{1-\iota\sqrt{3}}{2} \end{bmatrix}$$

4. Every matrix A such that $A^2 = A$ is similar to a diagonal matrix.

Solution. Given that $A^2 = A \Rightarrow A(A-I) = 0$

i.e. minimal polynomial will divide $x(x-1)$

$\Rightarrow$ minimal polynomial is exactly $x(x-1)$

$\because$ minimal polynomial has distinct factors, So A is diagonalizable i.e. similar to a diagonal matrix.

5. Let X be a finite dimensional complex inner product space and let T be operator on X. Show that T is self adjoint if and only if $\langle T\alpha, \alpha\rangle$ is real for every α in X.

Solution. Let T is self adjoint i.e. $T = T^*$ then,

$$\begin{aligned} \langle T\alpha, \alpha\rangle &= \langle \alpha, T^*\alpha\rangle \\ &= \langle \alpha, T\alpha\rangle \\ &= \overline{\langle T\alpha, \alpha\rangle} \end{aligned}$$

Conversely, $\langle T\alpha, \alpha\rangle$ is real.

$$
\begin{aligned}
\overline{\langle T\alpha, \alpha\rangle} &= \langle T\alpha, \alpha\rangle \\
&= \langle \alpha, T^*\alpha\rangle \\
\text{and } \overline{\langle T\alpha, \alpha\rangle} &= \overline{\langle \alpha, T^*\alpha\rangle} \\
&= \langle T^*\alpha, \alpha\rangle \\
&= \langle \alpha, T\alpha\rangle \ \forall \alpha \\
\text{So } \langle \alpha, T^*\alpha\rangle &= \langle \alpha, T\alpha\rangle \\
\Rightarrow \ \langle \alpha, (T^* - T)\alpha\rangle &= 0 \ \forall \alpha \\
\Rightarrow \ (T^* - T)\alpha &= 0 \ \forall \alpha \\
\Rightarrow \ T^* &= T
\end{aligned}
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