Self Evaluation Test

1. If $a_3 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$, $a_2 = \begin{pmatrix} -1 & 0 \\ 1 & 3 \end{pmatrix}$, $a_1 = \begin{pmatrix} 2 & 3 \\ -2 & 0 \end{pmatrix}$, $a_0 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ and $\mu_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\mu_0 = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ where $a(\theta) = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3$ $r(\theta) = r_0 + r_1(\theta)$ then the right quotient is $q(\theta) = q_0 + q_1(\theta) + q_2(\theta)^2$ where $q_2 = \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix}$, $a_1 = \begin{pmatrix} 7 & -14 \\ -2 & 5 \end{pmatrix}$, $a_0 = \begin{pmatrix} -43 & 81 \\ 12 & -24 \end{pmatrix}$ and the right remainder is $r_0 = \begin{pmatrix} 43 & -206 \\ -11 & 62 \end{pmatrix}$ on the right division of $a(\theta)$ by $r(\theta)$.

Solution. We have to prove that $q(\theta) = q_0 + q_1(\theta) + q_2(\theta)^2$ is the right quotient and $r(\theta) = r_0$ is the right remainder on the right division of $a(\theta)$ by $r(\theta)$.

i.e. $a(\theta) = q(\theta)\mu(\theta) + r(\theta)$ (1)

with $\deg r(\theta) < \deg \mu(\theta)$

$$\begin{aligned} \text{Given } \mu(\theta) &= \mu_0 + \mu_1(\theta) \\ &= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\theta \\ \text{and } a(\theta) &= a_0 + a_1(\theta) + a_2\theta^2 + a_3\theta^3 \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ -2 & 0 \end{pmatrix}\theta + \begin{pmatrix} -1 & 0 \\ 1 & 3 \end{pmatrix}\theta^2 + \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\theta^3 \end{aligned}$$

Now R.H.S. of equation (1) is

$$\begin{aligned} q(\theta)\mu(\theta) + r(\theta) &= (q_0 + q_1\theta + q_2\theta^2)(\mu_0 + \mu_1\theta) + r_0 \\ &= \left\{ \begin{pmatrix} -43 & 81 \\ 12 & -24 \end{pmatrix} + \begin{pmatrix} 7 & -14 \\ -2 & 5 \end{pmatrix}\theta + \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix}\theta^2 \right\} \cdot \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} + \right. \\ & \left. \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\theta \right\} + \begin{pmatrix} 43 & -206 \\ -11 & 62 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} -43 & 81 \\ 12 & -24 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} + \left\{ \begin{pmatrix} -43 & 81 \\ 12 & -24 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 7 & -14 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \right\} \theta + \left\{ \begin{pmatrix} 7 & -14 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \right\} \theta^2$$

$$+ \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \theta^3 + \begin{pmatrix} 43 & -206 \\ -11 & 62 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ -2 & 0 \end{pmatrix} \theta + \begin{pmatrix} -1 & 0 \\ 1 & 3 \end{pmatrix} \theta^2 + \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \theta^3$$

$$= a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3$$

$$= \text{L.H.S.}$$

$\therefore q(\theta)$ is right quotient and $r(\theta) = r_0$ is right remainder.

2. The ring of polynomials over a field is a Euclidean domain.

Solution. Let $f(x)$ be the ring of polynomials over a field F.

Let g be the function defined by:

$$g : F(x)/\{0\} \to \mathbb{N}$$

$$g[f(x)] = \text{deg} f(x) \text{ for all } f(x) \neq 0 \ \in F[x]$$

Thus we have assigned a non negative integer to every non zero element $f(x)$ in $F[x]$

Let $f(x)$ and $h(x)$ be two non zero polynomial and $k(x) = f(x)h(x)$ is also a non zero polynomial.

$$\begin{aligned} \text{Then } \deg(k(x)) &= \deg(f(x)h(x)) \\ &= \deg f(x) + \deg h(x) \\ \Rightarrow \deg(k(x)) &\geq \deg(f(x)) \ \ [\because \deg(h(x)) \geq 0] \\ \Rightarrow g(k(x)) &\geq g(f(x)) \end{aligned}$$

Again let $f(x) \ \in F[x]$ and $0 \neq h(x) \in F[x]$

There exist two polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = q(x)h(x) + r(x)$

where either $r(x) = 0$ or $\deg r(x) < \deg h(x)$ i.e.

either $r(x) = 0$ or $g(r(x)) < g(h(x))$.

Hence the ring of polynomials over a field is a Euclidean domain.

3. $Q[\sqrt{d}] := \{a + b\sqrt{d} | a, b \in Q\}$ is a field where $d \neq 0$ is a square free integer.

Solution. Let $a_1 + b_1\sqrt{d}$ and $a_2 + b_2\sqrt{d}$ both are the elements of $Q[\sqrt{d}]$. Then $a_1, b_1, a_2, b_2 \in Q$.

Now $(a_1+b_1\sqrt{d})+(a_2+b_2\sqrt{d})=(a_1+a_2)+(b_1+b_2)\sqrt{d}\ \in Q[\sqrt{d}]$ [$\because\ a_1+a_2\in Q$ and $b_1+b_2\in Q$]

also $(a_1+b_1\sqrt{d})+(a_2+b_2\sqrt{d})=(a_1a_2+db_1b_2)+(a_1b_2+a_2b_1)\sqrt{d}\ \in Q[\sqrt{d}]$

Since $a_1a_2+db_1b_2\ \in\ Q$ and $a_1b_2+a_2b_1\in Q$

$\Rightarrow\ Q[\sqrt{d}]$ is closed w.r.t addition and multiplication.

Here all the elements of $Q[\sqrt{d}]$ are real numbers and we know that addition and multiplication are both associative as well as commutative compositions in the set of real number.

Existence of identity:- $0+0\sqrt{d}\in Q[\sqrt{d}]$ since $0\in Q$.

Now if $a+b\sqrt{d}\ \in\ Q[\sqrt{d}]$ then $(0+0\sqrt{d})+(a+b\sqrt{d})=(0+a)+(0+b)\sqrt{d}=a+b\sqrt{d}$.

$\Rightarrow\ (0+0\sqrt{d})$ is identity.

Again if $a+b\sqrt{d}\ \in\ Q[\sqrt{d}]$ then $(-a)+(-b)\sqrt{d}\ \in\ Q[\sqrt{d}]$ and we have

$$((-a)+(-b)\sqrt{d})+(a+b\sqrt{d})=0+0\sqrt{d}$$

Therefore each element of $Q[\sqrt{d}]$ possess additive inverse.

Further in the set of real number, multiplication is distributive w.r.t. addition.

Again if $1+0\sqrt{d}\ \in\ Q[\sqrt{d}]$ we have

$$(1+0\sqrt{d})(a+b\sqrt{d})=a+b\sqrt{d}=(a+b\sqrt{d})(1+0\sqrt{d})$$

$\Rightarrow\ (1+0\sqrt{d})$ is multiplicative identity.

$\therefore\ Q[\sqrt{d}]$ is a commutative ring with unity.

Now $Q[\sqrt{d}]$ will be a field if each non zero element of $Q[\sqrt{d}]$ possesses multiplicative inverse.

Let $a+b\sqrt{d}$ be any non zero element of this ring. Then

$$\begin{aligned}\frac{1}{a+b\sqrt{d}} &= \frac{a-b\sqrt{d}}{a^2-db^2}\\ &= \frac{a}{a^2-db^2}+\left(\frac{-b}{a^2-db^2}\right)\sqrt{d}\end{aligned}$$

Now if $a,b\in Q$ then $a^2=db^2$ only if $a=0,\ b=0$

Since here atleast one of all the rational numbers $a\&\ b$ is not zero($\because\ a+b\sqrt{d}$ is a non zero element).

$\Rightarrow\ a^2\neq db^2$

$\therefore\ \dfrac{a}{a^2-db^2}$ and $\dfrac{-b}{a^2-db^2}$ are both rational numbers and atleast one of them is not zero.

$\therefore\ \left(\dfrac{a}{a^2-db^2}\right)+\left(\dfrac{-b}{a^2-db^2}\right)\sqrt{d}$ is a non-zero element of $Q[\sqrt{d}]$ and is multiplicative inverse of $a+b\sqrt{d}$.

Hence the given system is a field.

4. Let $K[\theta]$ be a ring of polynomials and let $a(\theta)$ and $b(\theta)$ be any two non zero elements of $K[\theta]$. Then

(a) $\deg[a(\theta)+b(\theta)]\ \leq\ \max[\deg a(\theta),\deg b(\theta)]=\max(n,m)$

(b) $\deg[a(\theta)b(\theta)] \ \leq \ \deg a(\theta) + \deg b(\theta)$ if $a(\theta)b(\theta) \ \neq 0$.

Solution. (a) Let $a(\theta) = a_0 + a_1\theta + a_2\theta^2 + \ldots + a_n\theta^n, \ a_n \neq 0$.

$b(\theta) = b_0 + b_1\theta + b_2\theta^2 + \ldots + b_m\theta^m, \ b_m \neq 0$ be two elements of $K[\theta]$

$\deg a(\theta) = n$ and $\deg b(\theta) = m$.

Now from the definition of sum of two polynomials, it is obvious that if $a(\theta) + b(\theta) \neq 0$ then,

$$\deg(a(\theta) + b(\theta)) = \begin{cases} \max(n, m), & \text{if } n \neq m; \\ n, & \text{if } n = m \text{ and } a_n + b_m \neq 0; \\ < n, & \text{if } n = m \text{ and } a_n + b_m = 0. \end{cases}$$

$\therefore \deg[a(\theta) + b(\theta)] \ \leq \ \max[\deg a(\theta), \deg b(\theta)] = \max(n, m)$.

(b) Again $a(\theta)b(\theta) = a_0b + (a_0b_1 + a_1b_0)\theta + \ldots + a_nb_m\theta^{n+m}$

If $a(\theta)b(\theta) \neq 0$ then $a(\theta)b(\theta)$ has unique degree.

If $a_nb_m \neq 0$ then $\deg[a(\theta)b(\theta)] = n + m = \deg a(\theta) + \deg b(\theta)$.

If $a_nb_m = 0$ then $\deg[a(\theta)b(\theta)] < n + m$.

$\therefore \deg[a(\theta)b(\theta)] \ \leq \ \deg a(\theta) + \deg b(\theta)$.

5. $K = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C} \right\}$, let $\alpha = a + \iota b$ and $\beta = c + \iota d$ where $a, b, c, d \ \in \mathbb{R}$.

$K = \left\{ \begin{pmatrix} a + \iota b & c + \iota d \\ -(c - \iota d) & a - \iota b \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$ is a skew field not a field.

Solution. Let $A = \begin{bmatrix} a + \iota b & c + \iota d \\ -c + \iota d & a - \iota b \end{bmatrix}$ and $B = \begin{bmatrix} p + \iota q & r + \iota s \\ -r + \iota s & p - \iota q \end{bmatrix}$

$$A + B = \begin{bmatrix} (a + p) + \iota(b + q) & (c + r) + \iota(d + s) \\ -(c + r) + \iota(d + s) & (a + p) - \iota(b + q) \end{bmatrix} \in K$$

$$\text{Also, } AB = \begin{bmatrix} (a + \iota b)(p + \iota q) + (c + \iota d)(-r + \iota s) & (a + \iota b)(r + \iota s) + (c + \iota d)(p - \iota q) \\ (-c + \iota d)(p + \iota q)) + (a - \iota b)(-r + \iota s) & (-c + \iota d)(r + \iota s)) + (a - \iota b)(p - \iota q) \end{bmatrix}$$

$$= \begin{bmatrix} (ap - bq - cr - ds) + \iota(aq + bp + cs - dr) & (ar - bs + cp + dq) + \iota(as + br - cq + dp) \\ -(cp + dq + ar - bs) + i(dp - cq + as + br) & (-cr - ds + ap - bq) - i(cs - dr + aq + bp) \end{bmatrix}$$

which is obviously an element of k.

$\therefore k$ is closed with respect to addition and multiplication.

Matrix addition is commutative as well as associative also.

Additive identity:- The zero matrix $\begin{bmatrix} 0+\iota 0 & 0+\iota 0 \\ -0+\iota 0 & 0-\iota 0 \end{bmatrix}$ is additive identity and so it is the zero element of k.

Additive inverse:- $A = \begin{bmatrix} a+\iota b & c+\iota d \\ -c+\iota d & a-\iota b \end{bmatrix} \in K$ then obviously $-A = \begin{bmatrix} -a-\iota b & -c-\iota d \\ c-\iota d & -a+\iota b \end{bmatrix} \in K$

$\Rightarrow$ each element of k possesses additive inverse.

Further matrix multiplication is associative and distributive with respect to addition.

$\Rightarrow$ k is a ring with respect to addition and multiplication of matrices.

Existence of multiplicative identity:-

$$\begin{bmatrix} 1+\iota 0 & 0+\iota 0 \\ -0+\iota 0 & 1-\iota 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in K$$

Thus K is a ring with unity.

Existence of multiplicative inverse of each non-zero element of K.

Let $A = \begin{bmatrix} a+\iota b & c+\iota d \\ -c+\iota d & a-\iota b \end{bmatrix} \in K$ be any non-zero element i.e. a, b, c, d are not all equal to zero.

$|A| = a^2 + b^2 + c^2 + d^2 \neq 0$

$\Rightarrow$ A is non singular and is therefore inveritble.

Now $A^{-1} = \frac{1}{|A|}\text{Adj}.A = \frac{1}{|A|}\begin{bmatrix} a-\iota b & -c+\iota d \\ c-\iota d & a+\iota b \end{bmatrix} \in K$ $\therefore$ K is a skew field. Now K is not a field i.e. multiplication is not commutative for example let $A = \begin{bmatrix} 3+4\iota & 5+6\iota \\ -5+\iota 6 & 3-\iota 4 \end{bmatrix} \in K$ and

$B = \begin{bmatrix} 1+\iota 0 & 1+\iota 0 \\ -1+\iota 0 & 1-\iota 0 \end{bmatrix} \in K$

then $AB = \begin{bmatrix} -2-2\iota & 8+10\iota \\ -8+10\iota & -2+2\iota \end{bmatrix}$ and $BA = \begin{bmatrix} -2+10\iota & 8+2\iota \\ -8+2\iota & -2-10\iota \end{bmatrix}$

$\therefore$ $AB \neq BA \Rightarrow$ M is not a field.