Topic 1

Modules

Throughout, R will denote an associate ring with identity $1 \neq 0$.

Definition 1. Let R be a ring. A left $R-$module is an additive abelian group M together with a function $R \times M \to M$, where (r, m) is mapped to rm, such that for every $r, s \in R$ and $m_1, m_2 \in M$:

$$
\begin{aligned}
(M1) \quad & r(m_1 + m_2) && = rm_1 + rm_2 \\
(M2) \quad & (r + s)m_1 && = rm_1 + sm_1 \\
(M3) \quad & r(sm_1) && = (rs)m_1 \\
(M4) \quad & 1.m_1 && = m_1, \text{ where 1 is the identity element of } R.
\end{aligned}
$$

A right R-module M is defined similarly via a function $M \times R \to R$ given by $(m, r) \to mr$ and satisfying obvious analogues of $(M1) - (M4)$. We will denote a left(right) R-module M by $_RM$. A module may be regarded as a generalization of vector space. The scalar multiplication in the vector space by field elements is replaced in a module by multiplication by arbitrary ring elements.

Note: From now on, unless otherwise stated, R-module means a left R-module. Also it is understood that all theorems which hold for left R-module, also hold in a similar way for right R-modules.

Let R be a commutative ring. Then it is easy to check that any left R-module is also a right R-module by defining $m.r = rm$. Hence for commutative rings, we do not distinguish between left and right R-modules.

Definition 2. Let R and S be rings. Then an abelian group M is called an (R, S)-bimodule if M is a left R-module as well as a right S-module such that the two scalar multiplication satisfy $r(ms) = (rm)s$. We will denote an (R, S)-module by $_RM_S$.

Suppose M is an R-module. Define a map θ from R to $End(M)$, the ring of all group endomorphisms of M, by $r \mapsto f_r$, where $f_r(m) = rm \ \forall m \in M$. Now $(f_r + f_s)(m) = rm + sm = (r + s)m = f_{r+s}(m)$ and $f_{rs}(m) = (rs)m = r(sm) = f_r f_s(m) \ \forall m \in M$ implies that θ is a ring homomorphism. In fact R-modules are completely determined by such ring homomorphisms. Suppose M is an abelian group and R is a ring such that there exists a ring homomorphism $\theta : R \longrightarrow End(M)$. Then my defining $rm = \theta(r)(m)$, M becomes an R-module.

Elementary properties of an R-module M:

(i) $0.m = 0 \;\; \forall \; m \in M$

(ii) $r.0 = 0 \;\; \forall \; r \in R$

(iii) $(-r)m = -(rm) = r(-m) \;\; \forall \; r \in R, \; m \in M.$

Here '0' written on the right side is the zero of M and 0 on the left side is the zero of R.

Proof. (i) $rm = (r + 0)m = rm + 0m \Rightarrow 0m = 0$

(ii) $rm = r(m + 0) = rm + r0 \Rightarrow r0 = 0$

(iii) $0 = 0m = (r + (-r))m = rm + (-r)m \Rightarrow (-r)m = -(rm)$

$0 = r.0 = r(m + (-m)) = rm + r(-m) \Rightarrow r(-m) = -(rm).$

□

Examples of Modules:

1. Let M be any additive abelian group. Then M is a left and a right $\mathbb{Z}$-module with respect to
   $n.m = m + m + \cdots + m$ (n-times)
   $-n.m = (-m) + (-m) + \cdots + (-m)$ (n-times).

2. Let $M_1, ..., M_n$ be R-modules and let $M = M_1 \times ... \times M_n$ be the cartesian product of $M_i's$. Then M admits a natural R-module structure with respect to addition and multiplication given by

   $$(x_1, ..., x_n) + (y_1, ..., y_n) = (x_1 + y_1, ..., x_n + y_n) \text{ and } r(x_1, ..., x_n) = (rx_1, ..., rx_n)$$

3. Let R be any ring. Then R is left as well as right R-module. For $r \in R, \; m \in R$ define rm and mr to be the product of r and m as elements of R. In fact R is an (R, R)-bimodule.

4. Every R-module M is a $\mathbb{Z}$-module, and hence is an $(R, \mathbb{Z})$-bimodule.

5. Let $M = \mathbb{M}_{m\times n}(R)$ = the set of all $m \times n$ matrices over a ring R. Then M becomes an R-module under the multiplication $r(a_{ij}) = (ra_{ij})\ \ \forall\, r \in R$. In particular, taking $m = 1,\ M = R^n$ is an R-module.

6. Let S be a ring and R be its subring. Then S is an R-module with respect to the usual product in S. In particular the rings $R[x_1, x_2, \cdots, x_n]$ and $R[[x]]$ are R-modules.

7. Let I be a left ideal of a ring R. Then I is a left R-module with respect to usual product in R. Furthermore, the quotient group (additive) R/I is an R-module with $r(s + I) = rs + I$.

8. Let A be an abelian group and let $\text{End}(A) = R$ be the endomorphism ring of A. Then A is an R-module with $fa = f(a),\ f \in R, a \in A$.

9. Let R and S be rings and $\theta : R \to S$ be a ring homomorphism. Then every S-module M can be made into an R-module by defining $rm = \theta(r)m$. It is said that the R-module structure of M is given by pullback along θ.