# Measure and integration on product spaces (Lectures 24, 25, 26, 27, 28, 29, 30, 31 and 32)

## 6.1. Product measure spaces

(6.1) Let $(X, \mathcal{A})$ be a measurable space. Let $\alpha, \beta \in \mathbb{R}$ and $E \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$. Show that $\{(x,t) \in X \times \mathbb{R} \,|\, (x, \alpha t + \beta) \in E\} \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$.
(Hint: Use the $\sigma$-algebra technique.)

(6.2) Let $E \in \mathcal{B}_{\mathbb{R}}$. Show that

$$\{(x,y) \in \mathbb{R}^2 \,|\, x + y \in E\}$$

and

$$\{(x,y) \in \mathbb{R}^2 \,|\, x - y \in E\}$$

are elements of $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.

(6.3) Let $X$ and $Y$ be nonempty sets and $\mathcal{C}$, $\mathcal{D}$ be nonempty families of subsets of $X$ and $Y$, respectively, as in proposition 7.1.5. Is it true that $\mathcal{S}(\mathcal{C} \times \mathcal{D}) = \mathcal{S}(\mathcal{C}) \otimes \mathcal{S}(\mathcal{D})$ in general? Check in the case when $\mathcal{C} = \{\emptyset\}$ and $\mathcal{D}$ is a $\sigma$-algebra of subsets of $Y$ containing at least four elements.

(6.4) Let $\mathcal{B}_{\mathbb{R}^2}$ denote the $\sigma$-algebra of Borel subsets of $\mathbb{R}^2$, i.e., the $\sigma$-algebra generated by the open subsets of $\mathbb{R}^2$. Show that

$$\mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}.$$

(Hint: Use proposition 7.1.5.)

### 6.2. Product of measure spaces

(6.5) For $E, F, E_i \in \mathcal{A} \otimes \mathcal{B}$ and $i \in I$, any indexing set, the following hold $\forall\, x \in X, y \in Y$ :

(i) $(\bigcup_{i\in I} E_i)_x = \bigcup_{i\in I}(E_i)_x$ and $(\bigcup_{i\in I} E_i)^y = \bigcup_{i\in I}(E_i)^y$.

(ii) $(\bigcap_{i\in I} E_i)_x = \bigcap_{i\in I}(E_i)_x$ and $(\bigcap_{i\in I} E_i)^y = \bigcap_{i\in I}(E_i)^y$.

(iii) $(E \setminus F)_x = E_x \setminus F_x$ and $(E \setminus F)^y = E^y \setminus F^y$.

(iv) If $E \subseteq F$, then $E_x \subseteq F_x$ and $E^y \subseteq F^y$.

(6.6) Let $E \in \mathcal{A} \otimes \mathcal{B}$ be such that $\mu(E^y) = 0$ for a.e. $(\nu) y \in Y$. Show that $\mu(E_x) = 0$ for a.e. $(\mu) x \in X$. What can you say about $(\mu \times \nu)(E)$?

### 6.3. Integration on product spaces: Fubini's theorems

(6.7) Let $f : X \times Y \longrightarrow \mathbb{R}$ be $\mathcal{A} \otimes \mathcal{B}$-measurable. Show that the following statements are equivalent:

(i) $f \in L_1(\mu \times \nu) := L_1(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$.

(ii) $\displaystyle\int_Y \left(\int_X |f(x,y)|\, d\mu(x)\right) d\nu(y) < +\infty.$

(iii) $\displaystyle\int_X \left(\int_Y |f(x,y)| d\nu(y)\right) d\mu(x) < +\infty.$

(6.8) Let $X = Y = [0,1], \mathcal{A} = \mathcal{B} = \mathcal{B}_{[0,1]}$, and let $\mu = \nu$ be the Lebesgue measure on [0,1]. Let

$$f(x,y) := \begin{cases} \dfrac{x^2 - y^2}{(x^2+y^2)^2} & \text{if} \quad (x,y) \neq (0,0), \\ 0 & \text{if} \quad x = y \text{ otherwise.} \end{cases}$$

Show that

$$\int_0^1 \left(\int_0^1 f(x,y) d\mu(x)\right) d\nu(y) \;=\; -\int_0^1 \left(\int_0^1 (x,y) d\nu(y)\right) d\mu(x) = -\pi/4.$$

This does not contradict, give reasons to justify.

(6.9) Let $X = Y = [-1,1], \mathcal{A} = \mathcal{B} = \mathcal{B}_{[-1,1]}$, and let $\mu = \nu$ be the Lebesgue measure on $[-1,1]$. Let

$$f(x,y) := \begin{cases} \dfrac{xy}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\int_{-1}^{1}\left(\int_{-1}^{1} f(x,y)d\nu(y)\right)d\mu(x) \;=\; 0 \;=\; \int_{-1}^{1}\left(\int_{-1}^{1} f(x,y)d\mu(x)\right)d\nu(y).$$

Can you conclude that

$$\int_{-1}^{1}\left(\int_{-1}^{1} f(x,y)d\nu(y)\right)d\mu(x) \;=\; \int_{X\times Y} f(x,y)d(\mu\times\nu)(x,y)?$$

(6.10) Let $f \in L_1(X,\mathcal{A},\mu)$ and $g \in L_1(Y,\mathcal{B},\nu)$. Let

$$\phi(x,y) \;:=\; f(x)g(y),\; x\in X \text{ and } y \in Y.$$

Show that $\phi \in L_1(X\times Y, \mathcal{A}\otimes\mathcal{B}, \mu\times\nu)$ and

$$\int_{X\times Y}\phi(x,y)d(\mu\times\nu) \;=\; \left(\int_X f d\mu\right)\left(\int_Y g d\nu\right).$$

(6.11) Let $f \in L_1(0,a)$ and let

$$g(x) := \int_x^a (f(t)/t)d\lambda(t), 0 < x \le a.$$

Show that $g \in L_1(0,a)$, and compute $\displaystyle\int_0^a g(x)d\lambda(x)$.

(6.12) Let $(X,\mathcal{A},\mu)$, and $(X,\mathcal{B},\nu)$ be as in exercise 6,8. Define, for $x,y \in [0,1]$,

$$f(x,y) := \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 2y & \text{if } y \text{ is irrational.}\end{cases}$$

Compute

$$\int_0^1\left(\int_0^1 f(x,y)d\nu(y)\right)d\mu(x) \;\text{ and }\; \int_0^1\left(\int_0^1 f(x,y)d\mu(x)\right)d\nu(y).$$

Is $f$ in $L_1(\mu\times\nu)$?

(6.13) Let $(X,\mathcal{A},\mu)$ be as in example 7.3.7. Let $Y = [1,\infty), \mathcal{B} = \mathcal{L}_{\mathbb{R}} \cap [1,\infty)$, and let $\nu$ be the Lebesgue measure restricted to $[1,\infty)$. Define, for $(x,y) \in X\times Y$,

$$f(x,y) := e^{-xy} - 2e^{-2xy}.$$

Show that $f \notin L_1(\mu\times\nu)$.

(6.14) Let $X$ be a topological space and let $\mathcal{B}_X$ be the $\sigma$-algebra of Borel subsets of $X$. A function $f : X \longrightarrow \mathbb{R}$ is said to be **Borel measurable** if $f^{-1}(E) \in \mathcal{B}_X \;\forall\, E \in \mathcal{B}_{\mathbb{R}}$. Prove the following:

(i) $f$ is Borel measurable iff $f^{-1}(U) \in \mathcal{B}_X$ for every open set $U \subseteq \mathbb{R}$. (Hint: Use the '$\sigma$-algebra technique'.)

(ii) Let $f : X \longrightarrow \mathbb{R}$ be continuous. Show that $f$ is Borel measurable.

(iii) Let $\{f_n\}_{n\geq 1}$ be a sequence of Borel measurable functions on $X$ such that $f(x) := \lim_{n\to\infty} f_n(x)$ exists $\forall\, x \in X$. Show that $f$ also is Borel measurable.

(iv) Consider $\mathbb{R}^2$ with the product topology and let $f, g$ be Borel measurable functions on $\mathbb{R}$. Show that the function $\phi$ on $\mathbb{R}^2$ defined by

$$\phi(x,y) := f(x)g(y), \quad x \in X, y \in Y,$$

is Borel measurable.

**Optional Exercises**

(6.15) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be Borel measurable. Show that for $x \in X$ fixed, $y \longmapsto f(x,y)$ is a Borel measurable function on $\mathbb{R}$. Is the function $x \longmapsto f(x,y)$, for $y \in Y$ fixed, also Borel measurable?

(6.16) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be such that for $x \in X$ fixed, $y \longmapsto f(x,y)$ is Borel measurable and for $y \in Y$ fixed, $x \longmapsto f(x,y)$ is continuous.

(i) For every $n \geq 1$ and $x, y \in \mathbb{R}$, define

$$f_n(x,y) := (i - nx)f((i-1)/n, y) + (nx - i + 1)f(i/n, y),$$

whenever $x \in [(i-1)/n, i/n), i \in \mathbb{Z}$. Show that each $f_n : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous and hence is Borel measurable.

(ii) Show that $f_n(x,y) \to f(x,y)$ as $n \to \infty$ for every $(x,y) \in \mathbb{R}^2$, and hence $f$ is Borel measurable.

(6.17) Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and let $f : X \times Y \longrightarrow \mathbb{R}$ be a nonnegative $\mathcal{A} \otimes \mathcal{B}$-measurable function. Let $\mu$ be a $\sigma$-finite measure on $(Y, \mathcal{B})$. For any $E \in \mathcal{B}$ and $x \in X$, let

$$\eta(x, E) := \int_E f(x,y)d\mu(y).$$

Show that $\eta(x, E)$ has the following properties:

(i) For every fixed $E \in \mathcal{B}, \longmapsto \eta(x, E)$ is an $\mathcal{A}$-measurable function.

(ii) For every fixed $x \in X, E \longmapsto \eta(x, E)$ is a measure on $(Y, \mathcal{B})$.

A function $\eta : X \times \mathcal{B} \longrightarrow [0, \infty)$ having properties (i) and (ii) above is called a **transition measure**.

## 6.4. Lebesgue measure on $\mathbb{R}^2$ and its properties

(6.18) Show that for $f \in L_1(\mathbb{R}^2, \mathcal{L}_{\mathbb{R}^2}, \lambda_{\mathbb{R}^2}), \boldsymbol{x} \in \mathbb{R}^2$, the function $\boldsymbol{y} \longmapsto f(\boldsymbol{x} + \boldsymbol{y})$ is integrable and

$$\int f(\boldsymbol{x} + \boldsymbol{y})d\lambda_{\mathbb{R}^2}(\boldsymbol{x}) = \int f(\boldsymbol{x})d\lambda_{\mathbb{R}^2}(\boldsymbol{x}).$$

(Hint: Use exercise 5.3.27 and theorem 7.4.3.)

(6.19) Let $E \in \mathcal{L}_{\mathbb{R}^2}$ and $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$. Let

$$\boldsymbol{x}E := \{(xt, yr) \mid (t, r) \in E\}.$$

Prove the following:

(i) $\boldsymbol{x}E \in \mathcal{L}_{\mathbb{R}^2}$ for every $\boldsymbol{x} \in \mathbb{R}, E \in \mathcal{L}_{\mathbb{R}^2}$, and $\lambda_{\mathbb{R}^2}(\boldsymbol{x}E) = |xy|\lambda_{\mathbb{R}^2}(E)$.

(ii) For every nonnegative Borel measurable function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$\int f(\boldsymbol{x}\boldsymbol{t})d\lambda_{\mathbb{R}^2}(\boldsymbol{t}) = |xy| \int f(\boldsymbol{t})d\lambda_{\mathbb{R}^2}(\boldsymbol{t}),$$

where for $\boldsymbol{x} = (x, y)$ and $\boldsymbol{t} = (s, r)$, $\boldsymbol{x}\boldsymbol{t} := (xs, yr)$.

(iii) Let $\lambda_{\mathbb{R}^2}\{\boldsymbol{x} \in \mathbb{R}^2 \mid |\boldsymbol{x}| \leq 1\} =: \pi$. Then

$$\lambda_{\mathbb{R}^2}\{\boldsymbol{x} \in \mathbb{R}^2 | \, |\boldsymbol{x}| < 1\} = \pi \quad \text{and} \quad \lambda_{\mathbb{R}^2}\{\boldsymbol{x} \in \mathbb{R}^2 | \, |\boldsymbol{x}| < r\} = \pi r^2.$$

(iv) Let $E$ be a vector subspace of $\mathbb{R}^2$. Then $\lambda_{\mathbb{R}^2}(E) = 0$ if $E$ has dimension less than 2.

(6.20) Let $T : \mathbb{R}^2 \to \mathbb{R}$ be a linear map.

(i) If $N \subseteq \mathbb{R}^2$ is such that $\lambda^*_{\mathbb{R}^2}(N) = 0$, show that $\lambda^*_{\mathbb{R}^2}(T(N)) = 0$.

(ii) Use (i) above and proposition 7.4.1(iii) to complete the proof of theorem 7.4.6 for sets $E \in \mathcal{L}_{\mathbb{R}^2}$.

(6.21) Consider the vectors $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$ and let

$$P := \{(\alpha_1 a_1 + \alpha_2 a_2, \alpha_1 b_1 + \alpha_2 b_2) \in \mathbb{R}^2 \mid \alpha_1, \alpha_2 \in \mathbb{R}, 0 \leq \alpha_i \leq 1\},$$

called the **parallelogram** determined by these vectors. Show that

$$\lambda_{\mathbb{R}^2}(P) = |a_1 b_2 - a_2 b_1|.$$