Measurable functions (Lectures 14, 15 and 16)

4.1. $\mathbb{L}_0$: Simple measurable functions

In this chapter, all the functions are defined on a measurable space $(X, \mathcal{S})$.

(4.1) Let $A, B \in \mathcal{S}$. Express the functions $|\chi_A - \chi_B|$ and $\chi_A + \chi_B - \chi_{A\cap B}$ as indicator functions of sets in $\mathcal{S}$ and hence deduce that they belong to $\mathbb{L}_0$.

(4.2) Let $s : X \longrightarrow \mathbb{R}^*$ be any function such that the range of s is a finite set. Show that $s \in \mathbb{L}_0$ iff $s^{-1}\{t\} \in \mathcal{S}$ for every $t \in \mathbb{R}^*$.

(4.3) Let $\{A_1, \ldots, A_n\}$ be subsets of X and

$$s = \sum_{i=1}^{n} a_i \chi_{A_i}.$$

Show that $s \in L_0$ iff each $A_i \in \mathcal{S}$.

(4.4) Let $s_1, s_2 \in \mathbb{L}_0$. Prove the following: Let $\forall\, x \in X$,

$$(s_1 \vee s_2)(x) := \max\{s_1(x), s_2(x)\} \text{ and } (s_1 \wedge s_2)(x) := \min\{s_1(x), s_2(x)\}.$$

Then $s_1 \wedge s_2$ and $s_1 \vee s_2 \in \mathbb{L}_0$.

(4.5) Express the functions $\chi_A \wedge \chi_B$ and $\chi_A \vee \chi_B$, for $A, B \in \mathcal{S}$, in terms of the functions χ_A and χ_B.

(4.6) Let $s_1, s_2 \in \mathbb{L}_0$ be real valued and $s_1 \geq s_2$. Let Show that $s_1 - s_2 \in \mathbb{L}_0$.

(4.7) Show that in general $\mathbb{L}_0^+$ need not be closed under limiting operations.

4.2. $\mathbb{L}$: Measurable functions

(4.8) Let $f : X \longrightarrow \mathbb{R}^*$ be a nonnegative measurable function. Show that there exist sequences of nonnegative simple functions $\{s_n\}_{n\geq 1}$ and $\{\tilde{s}_n\}_{n\geq 1}$ such that

$$0 \leq \cdots \leq s_n(n) \leq s_{n+1}(x) \leq \cdots \leq f(x) \leq \cdots \leq \tilde{s}_{n+1}(x) \leq \tilde{s}_n(x) \cdots$$

and $\lim_{n\to\infty} s_n(x) = f(x) = \lim_{n\to\infty} \tilde{s}_n(x) \ \forall\, x \in X.$

(4.9) Let f and $g : X \longrightarrow \mathbb{R}^*$ be measurable functions, p and $\alpha \in \mathbb{R}$ with $p > 1$, and let m be any positive integer. Use proposition 4.3.9to prove the following:

(i) $f + \alpha$ is a measurable function.

(ii) Let β and $\gamma \in \mathbb{R}^*$ be arbitrary. Define for $x \in \mathbb{R}$,

$$f^m(x) := \begin{cases} (f(x))^m & \text{if } f(x) \in \mathbb{R}, \\ \beta & \text{if } f(x) = +\infty, \\ \gamma & \text{if } f(x) = -\infty. \end{cases}$$

Then f^m is a measurable function.

(iii) Let $|f|^p$ be defined similarly to f^m, where p is a nonnegative real number. Then $|f|^p$ is a measurable function.

(iv) Let $\beta, \gamma, \delta \in \mathbb{R}^*$ be arbitrary. Define for $x \in \mathbb{R}$,

$$(1/f)(x) := \begin{cases} 1/f(x) & \text{if} \quad f(x) \notin \{0, +\infty, -\infty\}, \\ \beta & \text{if} \quad f(x) = 0, \\ \gamma & \text{if} \quad f(x) = -\infty, \\ \delta & \text{if} \quad f(x) = +\infty. \end{cases}$$

Then $1/f$ is a measurable function.

(v) Let $\beta \in \mathbb{R}^*$ be arbitrary and A be as in proposition 4.3.8. Define for $x \in \mathbb{R}$,

$$(fg)(x) := \begin{cases} f(x)g(x) & \text{if} \quad x \notin A, \\ \beta & \text{if} \quad x \in A. \end{cases}$$

Then fg is a measurable function.

(4.10) Let $f : X \to \mathbb{R}^*$ be $\mathcal{S}$-measurable. Show that $|f|$ is also $\mathcal{S}$-measurable. Give an example to show that the converse need not be true.

(4.11) Let $(X, \mathcal{S})$ be a measurable space such that for every function $f : X \longrightarrow \mathbb{R}$, f is $\mathcal{S}$-measurable iff $|f|$ is $\mathcal{S}$-measurable. Show that $\mathcal{S} = \mathcal{P}(X)$.

(4.12) Let $f_n \in \mathbb{L}$, $n = 1, 2, \ldots$. Show that the sets

$$\{x \in X \mid \{f_n(x)\}_n \text{ is convergent}\}$$

and

$$\{x \in X \mid \{f_n(x)\}_{n \geq 1} \text{is Cauchy}\}$$

belong to $\mathcal{S}$.