Exercises:

1. Recast the notion of homotopy of paths in terms of morphisms of the category $\mathbf{Top^2}$.
2. Define a binary operation on $\mathbb{Z} \times \mathbb{Z}$ as follows
$$(a, b) \cdot (c, d) = (a + c, b + (-1)^a d)$$
Show that this defines a group operation on $\mathbb{Z} \times \mathbb{Z}$ and this group is called the semi-direct product of $\mathbb{Z}$ with itself. The standard notation for this is $\mathbb{Z} \ltimes \mathbb{Z}$. Compute the inverse of (a, b), compute the conjugate of (a, b) by (c, d) and the commutator of two elements. Determine the commutator subgroup and the the abelianization of $\mathbb{Z} \ltimes \mathbb{Z}$.
3. A morphism $\phi \in$ Mor (X, Y) in a category is said to be an equivalence if there exists $\psi \in$ Mor (Y, X) such that $\phi \circ \psi =$ id $_Y$ and $\psi \circ \phi =$ id $_X$. In a category whose objects are sets and morphisms are maps, show that if $g \circ f$ is an equivalence for $f \in$ Mor (X, Y) and $g \in$ Mor (Y, Z) then g is surjective and f is injective.
4. We say a category $\mathcal{C}$ admits finite products if for every pair of objects U, V in $\mathcal{C}$ there exists an object W and a pair of morphisms $p : W \longrightarrow U$, $q : W \longrightarrow V$ such that the following property holds. For every pair of morphisms $f : Z \longrightarrow U$, $g : Z \longrightarrow V$ there exists a unique morphism $f \times g \in$ Mor (Z, W) such that
$$p \circ (f \times g) = f, \quad q \circ (f \times g) = g.$$
Show that the categories $\mathbf{Top}$, $\mathbf{Gr}$ and $\mathbf{AbGr}$ admit finite products and in fact the usual product of topological spaces/groups serve the purpose with p and q being the two projection maps.
5. Discuss arbitrary products in a category generalizing the preceding exercise and discuss the existence of arbitrary products in the categories **Top**, **Gr** and **AbGr**.

6. We say a category $\mathcal{C}$ admits finite coproducts if for every pair of objects U, V in $\mathcal{C}$ there exists an object W and a pair of morphisms $p : U \longrightarrow W$, $q : V \longrightarrow W$ such that the following property holds. For every pair of morphisms $f : U \longrightarrow Z$, $g : V \longrightarrow Z$ there exists a unique morphism $f \oplus g \in$ Mor (W, Z) such that

$$(f \oplus g) \circ p = f, \quad (f \oplus g) \circ q = g.$$

Show that the category $\mathbf{AbGr}$ admits finite coproducts and in fact the usual product of groups serves the purpose where the maps p and q are the canonical injections:

$p : G \longrightarrow G \times H, \quad q : H \longrightarrow G \times H$

$p(g) = (g, 1), \quad q(h) = (1, h)$

What happens when this (naive construction) is tried out in the category $\mathbf{Gr}$ instead of $\mathbf{AbGr}$? In the context of abelian groups the coproduct is referred to as the direct sum.

7.

Discuss the coproduct of an arbitrary family of objects in the category **AbGr**. It is referred to as the direct sum of the family.

Suppose that X and Y are two topological spaces, form their disjoint union $X \sqcup Y$ which is the set theoretic union of their homeomorphic copies $X \times \{1\}$ and $Y \times \{2\}$.

A subset G of $X \sqcup Y$ is declared open if $G \cap (X \times \{1\})$ and $G \cap (Y \times \{2\})$ are both open. Check that this defines a topology on $X \sqcup Y$ and the maps

$p : X \longrightarrow X \sqcup Y, \quad q : Y \longrightarrow X \sqcup Y$

$p(x) = (x, 1), \quad q(y) = (y, 2)$

are both continuous. Show that the category $\mathbf{Top}$ admits finite coproducts.