Exercises:

1. Show that the maps i_1 and i_2 in definition (23.1) are injective and that the images of i_1 and i_2 generate $G_1 * G_2$. Hint: Use the universal property with $H = G_1,\ f_1 = i_1$ and $i_2 = 1$.
2. Show that abelianizing a free group on k generators results in a group isomorphic to the direct sum of k copies of $\mathbb{Z}$. Use the fact that the coproduct in **AbGr** is the direct sum.
3. Is there a surjective group homomorphism from the direct sum $\mathbb{Z} \times \mathbb{Z}$ onto $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$? Prove that if k and l are distinct positive integers, the free group on k generators is not isomorphic to the free group on l generators.
4. Show that $\langle a, c \mid a^2c^2 = 1\rangle$ is also a presentation of the fundamental group of the Klein's bottle.
5. Construct the push-out for the pair $j_1 : C \longrightarrow A_1$ and $j_2 : C \longrightarrow A_2$ in the category **AbGr**?
6. Suppose that C is the trivial group in the definition of push-out in the category **Gr**, show that the resulting group is the coproduct of the two given groups. What happens in the category **AbGr**? Describe explicitly the construction of the group specifying the subgroup that is being factored out.