Problem set 8 : Fundamental Theorem of Galois Theory

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(1) Let K be a splitting field of $x^4 - 2$ over $\mathbb{Q}$. List all elements of $G = G(K/\mathbb{Q})$. Draw a diagram showing primitive elements of all the subfields of $K/\mathbb{Q}$. Draw the lattice of the subgroups of G and match them with the fixed fields.

(2) Determine the Galois group of $f(x) = (x^2 - 2)(x^2 - 3)(x^2 - 5)$. Determine all the subfields of the splitting field of $f(x)$.

(3) Prove that the Galois group of $x^p - 2$, where p is a prime, is isomorphic to the group

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{F}_p \text{ and } a \neq 0 \right\}.$$

(4) Let $f(x) \in \mathbb{Z}[x]$ be an irreducible quartic with Galois group S_4 over $\mathbb{Q}$. Let θ be a root of $f(x)$. Show that there is no field properly contained in $\mathbb{Q}(\theta)/\mathbb{Q}$. Is $\mathbb{Q}(\theta)/\mathbb{Q}$ a Galois extension ?

(5) Show that if the Galois group of a rational cubic $f(x)$ is cyclic of order 3 then $f(x)$ has only real roots.

(6) Consider the polynomial $f(x) = x^4 - 2x^2 - 2$.

(a) Show that the roots of the quartic are

$$\alpha_1 = \sqrt{1 + \sqrt{3}},\ \alpha_2 = \sqrt{1 - \sqrt{3}},\ \alpha_3 = -\alpha_1 \text{ and } \alpha_4 = -\alpha_2.$$

(b) Prove that $K_1 = \mathbb{Q}(\alpha_1) \neq K_2 = \mathbb{Q}(\alpha_2)$ and $K_1 \cap K_2 = \mathbb{Q}(\sqrt{3}) = F$.

(c) Show that K_1, K_2 and K_1K_2 are Galois over F

(d) Show that $G(K_1K_2/F)$ is the Klein 4-group. Determine the automorphisms in this group.

(e) Show that the Galois group of $f(x)$ over $\mathbb{Q}$ is dihedral of order 8.

(7) Let $\mathbb{C}(X)$ denote the rational function field in the indeterminate X over $\mathbb{C}$. Let $a \in \mathbb{C}$ and $\sigma_a : \mathbb{C}(X) \to \mathbb{C}(X)$ be the automorphism that substitutes X by $X + a$. Put $G = \{\sigma_a : a \in \mathbb{C}\}$. Show that $\mathbb{C}(X)^G = \mathbb{C}$.

(8) Suppose that the Galois group of a field extension K/F is the Klein 4-group V_4. Show that K/F is biquadratic.

(9) Let $E = \mathbb{Q}(r)$ where r is a root of $f(x) = x^3 + x^2 - 2x - 1$ in $\mathbb{C}$. Show that $f(r^2 - 2) = 0$. Determine $G(E/\mathbb{Q})$.

(10) Let $E = \mathbb{C}(t)$ where t is a transcendental over $\mathbb{C}$. Let $\omega = e^{2\pi i/3}$. Define the $\mathbb{C}$-automorphisms σ and τ of E by the equations $\sigma(t) = \omega t$ and $\tau(t) = 1/t$. Show that

$$\sigma^3 = \tau^2 = id \text{ and } \tau\sigma = \sigma^{-1}\tau.$$

Show that the group G of automorphisms generated by σ and τ has order 6 and $E^G = \mathbb{C}(t^3 + t^{-3})$.

(11) Let x, y be variables. Let $a, b, c, d \in \mathbb{Z}$ and $n = |ad - bc|$. Show that $L = \mathbb{C}(x, y)$ is a Galois extension of $K = \mathbb{C}(x^a y^b, x^c y^d)$ of degree n. Find $G(L/K)$.