Problem set 7 : Primitive elements

---

(1) Let $\alpha = \sqrt[3]{2}, \zeta = (-1+\sqrt{-3})/2$ and $\beta = \alpha\zeta$.

(a) Prove that for all $c \in \mathbb{Q}, \gamma = \alpha + c\beta$ is a root of a sextic of the form $x^6 + ax^3 + b$.

(b) Prove that $\text{irr}(\alpha+\beta, \mathbb{Q})$ is cubic.

(c) Prove that $\text{irr}(\alpha-\beta, \mathbb{Q})$ is sextic.

(2) Let $\alpha = \sqrt[3]{2}$, and $\omega = e^{2\pi i/3}$. Show that $\omega + c\alpha$ is a primitive element of $\mathbb{Q}(\alpha, \omega)$ for all $c \in \mathbb{Q}^\times$.

(3) Let $\omega = e^{2\pi i/3}$. Show that $\omega\sqrt{5}$ is a primitive element of $\mathbb{Q}(\omega, \sqrt{5})$.

(4) Let F be a subfield of $\mathbb{C}$ and $a, b \in \mathbb{C}$ be algebraic elements over F. Show that there exist an integer n such that $a + nb$ is a primitive element of the field $K = F(a, b)$.

(5) Find infinitely many primitive elements of the field $\mathbb{Q}(a, \omega)$ where a is a root of $x^3 - x + 1$.

(6) Construct infinitely many intermediate subfields of $\mathbb{F}_p(u,v)/\mathbb{F}_p(u^p, v^p)$ where u, v are indeterminates.

(7) Find a primitive element of $\mathbb{F}_{2^4}$ over $\mathbb{F}_2$.

(8) Find a primitive element of the field extension $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.

(9) Let K/F be a finite separable extension of degree n. Using the primitive element theorem show that there are exactly n distinct embeddings of K into an algebraic closure F^a of F.