Problem set 5 : Separable Extensions

Notation: Throughout these exercises, $F \subset K \subset L$ is a tower of fields. Assume that char $F = p > 0$ in the problems 4-10.

(1) Let char $F = 0$ and $f(x) \in F[x]$ be a monic polynomial of positive degree. Let $d(x) = (f(x), f'(x))$. Show that $g(x) = f(x)/d(x)$ has same roots as $f(x)$ and $g(x)$ is separable.

(2) Let $a \in L$ be separable over F. Show that a is separable over K.

(3) Show that an algebraic extension of a perfect field is perfect.

(4) Let $f(x) = x^{p^n} - a \in F[x]$ where n is a positive integer. Show that $f(x)$ is irreducible over F if and only if $a \notin F^p$.

(5) Let $([K : F], p) = 1$. Show that K is a separable algebraic extension of F.

(6) Show that $\cap_{i=0}^{\infty} F^{p^i}$ is the largest perfect subfield of F.

(7) Let $f(x) \in F[x]$ be irreducible. Show that there exists an irreducible separable polynomial $g(x) \in F[x]$ and a positive integer e such that $f(x) = g(x^{p^e})$. Show that all roots of $f(x)$ have same multiplicity p^e.

(8) A polynomial $f(x) \in F[x]$ is called a $p-$*polynomial* if it is of the form $x^{p^m} + a_1 x^{p^{m-1}} + \cdots + a_m x$. Show that a monic polynomial of positive degree is a p-polynomial if and only if its roots form a finite subgroup of the additive group of a splitting field of $f(x)$ over F and every root has same multiplicity p^e.

(9) Let t be an indeterminate. Show that the field extension $F(t)/F(t^p)$ is not separable.

(10) Let $K = \mathbb{F}_p(t, w)$ be the rational function field in two indeterminates t, w over $\mathbb{F}_p$. Let L be the splitting field over K of the polynomial $h(x) = f(x)g(x)$ where $f(x) = x^p - t$ and $g(x) = x^p - w$. Prove the following:
(a) f and g are irreducible over K.
(b) $[L : K] = p^2$.
(c) L/K is not separable.
(d) $a^p \in K$ for all $a \in L$.