**Problem Set 12 : Cyclic Extensions**

---

(1) Let $E/F$ be a finite extension of finite fields. Show that $N_{E/F} : E^\times \to F^\times$ is surjective.

(2) Show that a nonzero $a \in \mathbb{Q}$ is norm of an element in $\mathbb{Q}(\sqrt{-1})$ if and only if the odd primes occurring with odd multiplicities in the numerator or denominator of $a$ written in reduced form are of the form $4n+1$.

(3) Let $p$ be a prime. Let $F$ is a field having $p$ distinct $p^{th}$ roots of unity and let $z \in F$ be a primitive $p^{th}$ root of unity. Let $E/F$ be cyclic of degree $p^r$. Show that if $E/F$ can be embedded in a cyclic field $K/F$ of degree $p^{r+1}$ then $z = N_{E/F}(u)$ for some $u \in E$.

(4) Show that if $m$ is a negative integer then $E = \mathbb{Q}(\sqrt{m})$ cannot be embedded in a cyclic quartic extension field over $\mathbb{Q}$.

(5) Let $K = \mathbb{Q}(\sqrt[n]{a})$ where $a \in \mathbb{Q}$ and $a > 0$. Let $[K : \mathbb{Q}] = n$. Let $E$ be a subfield of $K/\mathbb{Q}$ and $[E : K] = d$. Consider $N_{K/E}(\sqrt[n]{a})$ and show that $E = \mathbb{Q}(\sqrt[d]{a})$.

(6) Let $p$ be a prime and $K = \mathbb{Q}(z)$ where $z$ is a primitive $p^{th}$ root of unity. Let $G = G(K/\mathbb{Q})$. Let $w$ be any $p^{th}$ root of unity. Show that $Tr_{K/\mathbb{Q}}(w) = -1$ or $p-1$ depending on whether $w$ is or is not a primitive $p^{th}$ root of unity.