Problem Set 11 : Galois groups of Quartic Polynomials

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(1) Show that the resolvent cubic of the quartic polynomial $x^4 + px^2 + qx + r$ is $x^3 - px^2 - 4rx + (4pr - q^2)$.

(2) Determine the Galois groups of the quartics: $x^4 - 2$, $x^4 + 2$, $x^4 - x + 1$, $x^4 + x + 1, x^4 + x^3 + x^2 + x + 1$, and $x^4 + 4x^2 - 5$.

(3) Show that the resolvent cubic $r(x)$ of $f(x) = x^4 + 1$ is $x(x-2)(x+2)$ and $G_f = V$.

(4) Show that the Galois group of $e_4(x) = 1 + x + x^2/2 + x^3/3 + x^4/4$ is A_4.

(5) Show that the Galois group of an irreducible quartic in $\mathbb{Q}[x]$ with exactly two real roots is either S_4 or D_4.

(6) Let α be a real root of an irreducible rational quartic whose resolvent cubic is irreducible. Show that α is not constructible by ruler and compass. Can we construct the roots of the quartic $x^4 + x - 5$ by ruler and compass ?

(7) Put $f(x) = x^4 - 2x^2 - 1$. Show that $f(x)$ is irreducible in over $\mathbb{Q}$. and $r(x) = (x+2)(x^2+4)$. Show that $G_f = D_4$. Find all the intermediate subfields of the splitting field K of $f(x)$ over $\mathbb{Q}$. Match them with the subgroups of D_4.

(8) Show that the discriminants of a quartic polynomial and its resolvent cubic are equal.

(9) Substitute x by $1/y$ to calculate $\mathrm{disc}(x^4 + ax^3 + b)$.

(10) Find sufficient conditions on the integers a, b and c so that $\mathbb{Q}(\sqrt{a + b\sqrt{c}})$ is a Galois extension of $\mathbb{Q}$ with cyclic Galois group of order 4.