Problem set 10 : Solvability by Radicals

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(1) Show that the polynomials $f(x) = x^5 - 14x + 7$, $g(x) = x^5 - 7x^2 + 7$ $h(x) = x^7 - 10x^5 + 15x + 5$ and $\ell(x) = x^5 - 6x + 3$ are not solvable by radicals over $\mathbb{Q}$.

(2) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of prime degree p. Suppose that $f(x)$ has exactly two non-real roots. Show that $f(x)$ is not solvable by radicals over $\mathbb{Q}$.

(3) Let a be a positive rational number and $K = \mathbb{Q}(\sqrt[n]{a})$. Show that if n is odd then K has no notrivial subfield which is Galois over $\mathbb{Q}$. If n is even, show that the only nontrivial subfield of K that is Galois over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt{a})$.

(4) Let $F = \mathbb{F}_p$ and $K = F(x)$ be the function field in one variable x. Show that $f(x) = t^p - t - x \in K[t]$ is irreducible over K. Show that the Galois group of $f(x)$ over K is cyclic of order p. Is $f(x)$ solvable by radicals over K?

(5) Let K be a subfield of $\mathbb{C}$. Let $p(x) = x^3 + px + q$ be an irreducible polynomial in $K[x]$. Let r be a root of $p(x)$. Let $u = a + br + cr^2 \in K(r) \setminus K$. Determine $g(x) := \operatorname{irr}(u, K)$. Let $\Delta = -4p^3 - 27q^2$. Show that $K(r)$ is a radical extension of K if and only if -3Δ is a square in K.

(6) Let x_1, x_2, x_3 be indeterminates and let s_1, s_2, s_3 be the elementary symmetric polynomials of x_1, x_2, x_3. Show that $\mathbb{Q}(x_1, x_2, x_3)$ is not a radical extension of $\mathbb{Q}(s_1, s_2, s_3)$ but $\mathbb{Q}(\zeta_3)(x_1, x_2, x_3)$ is a radical extension of $\mathbb{Q}(s_1, s_2, s_3)$.

(7) Let G be the Galois group of an irreducible quintic over $\mathbb{Q}$. Show that $G = A_5$ or S_5 if G has an element of order 3.

(8) Is every Galois extension of degree 10 solvable by radicals ?

(9) Let ζ be a primitive 7^{th} root of unity and let $\alpha = \zeta + \zeta^{-1}$. Show that $f(x) = \operatorname{irr}(\alpha, \mathbb{Q}) = x^3 + x^2 - 2x - 1$. Solve for the roots of $f(x)$ to express ζ in terms of radicals over $\mathbb{Q}$.