Problem Set 1 : Algebraic Extensions

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(1) Let F be a finite field with characteristic p. Prove that $|F| = p^n$ for some n.

(2) Using $f(x) = x^2 + x - 1$ and $g(x) = x^3 - x + 1$, construct finite fields containing $4, 8, 9, 27$ elements. Write down multiplication tables for the fields with 4 and 9 elements and verify that the multiplicative groups of these fields are cyclic.

(3) Determine irreducible monic polynomials over $\mathbb{Q}$ for $1 + i$, $2 + \sqrt{3}$, and $1 + \sqrt[3]{2} + \sqrt[3]{4}$.

(4) Prove that $x^3 - 2$ and $x^3 - 3$ are irreducible over $\mathbb{Q}(i)$.

(5) Prove that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Find an irreducible polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.

(6) Determine the degree $[\mathbb{Q}(\sqrt{3 + 2\sqrt{2}}) : \mathbb{Q}]$.

(7) Prove that if $[F(\alpha) : F]$ is odd then $F(\alpha) = F(\alpha^2)$.

(8) Let K/F be an algebraic field extension and R be a ring such that $F \subset R \subset K$. Show that R is a field.

(9) Let K/F be an extension of degree n.

(a) For any $a \in K$, prove that the map $\mu_a : K \to K$ defined by $\mu_a(x) = ax$ for all $x \in K$, is a linear transformation of the F-vector space K. Show that K is isomorphic to a subfield of the ring $F^{n\times n}$ of $n \times n$ matrices with entries in F.

(b) Prove that a is a root of the characteristic polynomial of μ_a. Use this procedure to find monic polynomials satisfied by $\sqrt[3]{2}$ and $1 + \sqrt[3]{2} + \sqrt[3]{4}$.

(10) Let $K = \mathbb{Q}(\sqrt{d})$ for some square free integer d. Let $\alpha = a+b\sqrt{d} \in K$. Use the basis $B = \{1, \sqrt{d}\}$ of K over F and find the matrix $M_B^B(\mu_\alpha)$ of $\mu_\alpha : K \to K$ with respect to B. Prove directly that the map $a + b\sqrt{d} \mapsto M_B^B(\mu_\alpha)$, is an isomorphism of fields.

(11) Prove that -1 is not a sum of squares in the field $\mathbb{Q}(\beta)$ where $\beta = \sqrt[3]{2}\, e^{2\pi i/3}$.

(12) Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \in \mathbb{Z}[x]$. Suppose that $f(0)$ and $f(1)$ are odd integers. Show that $f(x)$ has no integer roots.

(13) Let R be an integral domain containing $\mathbb{C}$. Suppose that R is a finite dimensional $\mathbb{C}$-vector space. Show that $R = \mathbb{C}$.

(14) Let k be a field and x be an indeterminate. Let $y = x^3/(x+1)$. Find the minimal polynomial of x over $k(y)$.

(15) Find an algebraic extension K of $\mathbb{Q}(x)$ such that the polynomial

$$f(y) = y^2 - x^3/(x^2+1) \in \mathbb{Q}(x)[y]$$

has a root in K.