

## Arrangements and levels

1. A partition of  $S$  into two subsets  $S_1$  and  $S_2$  is called *linearly separable* if there is a line such that  $S_1$  lies on one side of the line and  $S_2$  on the other side.

- (a) Give a tight bound on the number of linearly separable partitions of a set of  $n$  points in the plane.

Solution: In the dual space, look the arrangements of lines (corresponding to the input points). All the points in a face correspond to a partitioning line in the original space corresponding to the same partition. Since there are  $O(n^2)$  faces there are  $O(n^2)$  distinct partitions. This bound is achievable as one can argue using a convex  $n - gon$ .

- (b) Let  $R$  and  $B$  denote two sets having  $m$  and  $n$  points respectively. If  $R$  and  $B$  are linearly separable, show that  $R$  and  $B$  can be simultaneously bisected by a single line. You may assume that  $n$  and  $m$  are even integers.

Hint: You may assume the line separating  $R$  and  $B$  to be the  $y$ -axis wlog and use duality. Solution: Consider the duals of the set  $B$  and  $R$  respectively - denote them by  $B^*$  and  $R^*$  respectively. The points in between the  $n/2$  and  $n/2 + 1$  levels in  $B^*$  correspond to the *halving partitions* of  $B$  - denote this by  $L_B$ . For the analogous points in  $R$ , denote this by  $L_R$ . The crux of the problem is to show that  $L_B$  and  $R_B$  intersect, whose dual (a line) simultaneously halves  $B$  and  $R$ .

For this, note that the lines corresponding to  $B$  and  $R$  have negative and positive slopes respectively. Now you can argue that two levels, one of which consists of lines with negative slopes and the other one consists of positively sloping lines must intersect.