Problems

1. Most phytoplankton in lakes are too small to be individually seen with the unaided eye. However, when present in high enough numbers, they may appear as a green discoloration of the water due to the presence of chlorophyll within their cells. Climate and water quality are among the factors influencing the quantity of phytoplankton in shallow lakes. Assume that the rate of increase of phytoplankton can be expressed as a linear function, g(X1, X2, X3) of three variables, namely X1 temperature of water, X2 global radiations and X3 concentrations of nutrients. X1, X2, X3 can be modeled as normal random variables. Positive growth rates must be avoided.

Although it is observed that temperature and radiation have no effect on effect on concentration of nutrients, so that ρ13 = ρ23 = 0, mutually they are highly correlated with ρ12 = 0.8. The equilibrium function is given by g(X1, X2, X3) = a0 + a1*X1 + a2* X2 + a3 *X3,

where a0 = -1.5 mg/cubic m, a1 = 0.08 mg/ (cubic m. degree Celsius), a2 = 0.01 mg/mW and a3 = 0.05

Solution.

Equilibrium (limit state), g(X1, X2, X3) = 0

RANDOM VARIABLE,X	Mean ,μ	Coefficient of Variation, V	Standard Deviation, σ
X1,degree Celsius	16	0.5	8
X2,W/square M	150	0.3	45
X3,mg/cubic m	100	0.7	70

Other variables are included in a0 because of difficulty in computing separately.

Reliability index $\beta = a0 + \sum ai * \mu i \Big/ \sqrt{\sum ai^2 \sigma i^2}$

$$\beta = 6.28 \Big/ \sqrt{13.32} = 1.72$$

$$\gamma = \Phi(1.72) = 0.957$$

There is a chance that the equilibrium situation 96 percent. Hence the risk that algal biomass will increase is only 4 percent.

2. The economic performance of the irrigation barrage located at a place in along a river could be improved by installing a hydropower station to meet the local energy demand. An engineer estimates the power demand X3 to be 600kW on average with variability µ 600kw. If standard turbo axial turbine was installed, power output can be estimated as 7.5X1X2.Discharge X1 is measured in cubic m/s and hydraulic head in m;7.5 is coefficient accounting for gravity, density of water, and overall efficiency of installed equipment. Accordingly, power is given in units of kW. Although average discharge of 22 cubic m/s and an average head of 5.2m are available, discharge head availability depends on natural flow variability; it is also subjected to the construction of barrage handling, which is operated with priority for irrigation demand. Discharge and head can be assumed to be independent normal variables, X1 and X2, with coefficients of variation 0.2 and 0.15 respectively. Assuming that demand X3 normal and independent of discharge and head, evaluate that reliability of the plant.

Performance function g(X1, X2, X3) = 7.5*X1*X2 – X3

RANDOM VARIABLE	UNIT	MEAN	COEFFICIENT OF VARIATION	STANDARD DEVAITAION
Normal Discharge X1	Cubic m/s	22	0.22	4.4
Normal Hydraulic Head X2	m	5.2	0.15	0.78
Normal Power Demand X3	kW	600	0.10	60

Step1.Partial differentiation of performance functions with respect to each random variable.

$$\left(\partial g / \partial x1\right) f = 7.5 * x2 * \sigma 1$$

$$\left(\partial g / \partial x2\right) f = 7.5 * x1 * \sigma 2$$

$$\left(\partial g / \partial x3\right) f = -\sigma 3$$

Step2.computation of direction cosines, α **.**

$$\alpha = \frac{\left(\partial g / \partial Xi\right) f}{\sqrt[2]{\left(\Sigma \left(\partial g / \partial Xi\right) * \left(\partial g / \partial Xi\right)\right)}}$$

Step3.calcualtion of new X_{if}

$X_{if} = \alpha \ \beta$

Step4.using estimated values of mean and standard deviation, calculate $X_{if\ new}$

$X_{ifnew} = \mu_i + \sigma \ X_{if}$

Step5.repeat the iteration until reliability index value converge to single value.

Evaluation of reliability

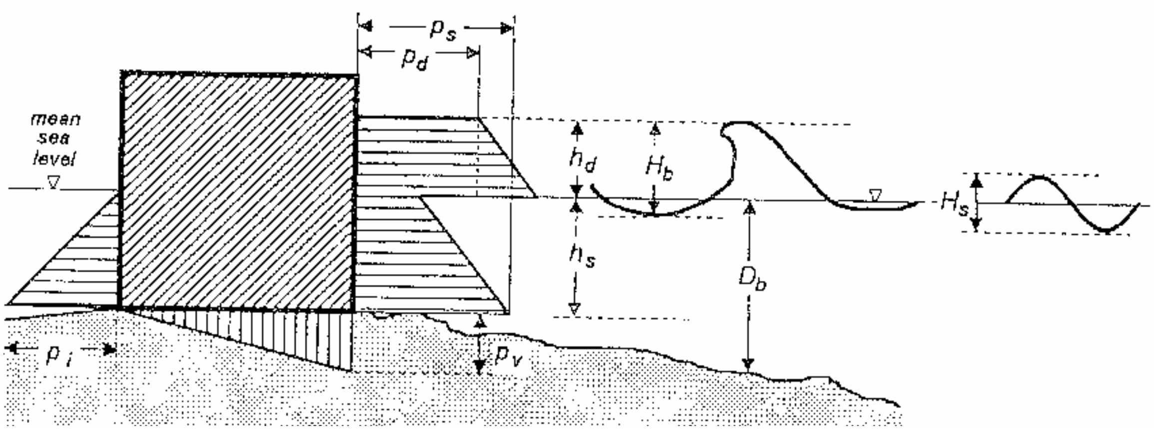
Limiting state of interest g(X1, X2, X3) = 7.5*X1*X2 – X3 = 0. Iteration process is illustrated in the following sections.

		MEAN	COEFFICINET OF VARIATION	STANDARD DEVIATION
INITIAL x_{1f}	22.0	17.8	17.7	17.7
INITIAL x_{2f}	5.2	4.6	4.7	4.7
INITIAL X_{3f}	600	620	623	623
$\left(\partial g / \partial X1\right) f$	171.6	153.2	154.6	154.8

$\left(\partial g / \partial X2\right) f$	128.7	104.2	103.7	103.6
$\left(\partial g / \partial X3\right) f$	-60.0	-60.0	-60.0	-60.0
$\Sigma\left(\partial g / \partial Xi\right) f$	49,610	37,922	38,254	38,276
$\alpha 1 f$	0.770	0.787	0.790	0.791
$\alpha 2 f$	0.578	0.535	0.530	0.529
$\alpha 3 f$	-0.269	-0.308	-0.307	-0.307
NEW x_{1f}	17.8	17.7	17.7	17.7
NEW x_{2f}	4.6	4.7	4.7	4.7
NEW X_{3f}	620	622.8	622.7	622.7
G(.) $7.5\ x_{1f} * x_{2f} - X_{3f}$	-4.5*10e-5	7.5*10e-6	1.7*10e-5	1.8*10e-5
β	1.24	1.23	1.23	1.23

It can be noted that last two iterations give identical value. This corresponds to reliability of 89% and the risk of failure of 11%.

3. Consider a harbor breakwater constructed with massive concrete tanks filled with sand. It is necessary to evaluate the risk that the breakwater will slide under pressure of a large wave during major storm.



The following information/data is necessary for analysis.

Resultant horizontal force, $\mathbf{R_h}$, depends on the balance between the static and dynamic pressure components, and it can be taken as quadratic function of H_b (indicated in Figure) under simplified hypothesis on the depth of the breakwater.

Random deep water value $\mathbf{X_4 = H_s}$, which is found from frequency analysis of extreme storms in the area.

Resultant vertical force, $\mathbf{R_v = X_2 - F_V}$

Where $\mathbf{X_2}$, weight of the tank reduced for buoyancy.

$\mathbf{F_V}$, a vertical component of dynamic uplift pressure due to the braking wave. It is proportional to height of the height of the design wave, H_b, when the slope of sea bottom is known.

Coefficient of friction, $\mathbf{c_f}$, can interpret as a random variable, $\mathbf{X_1}$, which represents inherent uncertainty associated with its field evaluation.

$if\ {}^{Rh}\!/_{Rv} < c_f$, stability against sliding will exist.

Additional variate $\mathbf{X_3}$ is introduced to represent the uncertainties caused the simplifications adopted to model the dynamic forces F_V and R_h .

Simplification of the shoaling effects indicates that the height H_b of the design wave is proportional to random deepwater value X_4.

All random variables are assumed to be independent.

The constants a_1, a_2, a_3 are depends on geometry of system.

Accounting for the sea-bottom profile and the geometry, one estimate constants, a_1=7, a_2 =17m/KN, a_3=145.

Limiting state equation

$$g(X_1, X_2, X_3.X_4) = X_1X_2 - 70X_1X_3X_4 - 17X_3X_4 - 17X_3X_4X_4 - 145X_3X_4 = 0 \qquad (1)$$

Random variables	Mean	Coefficient of variation	Standard deviation
X_1	0.64	0.15	0.096
X_2	3400 KiloNewton/m	0.05	108.80
X_3	1	0.20	0.2
X_4	5.16	0.18	0.93

Partial derivatives of performance function with respect to each random variable evaluated at failure point.

$$\left(\partial g / \partial X1\right)_f = (x_2 - 70x_3 * x_4)\,\sigma_1$$

$$\left(\partial g / \partial X2\right)_f = x_1 * \sigma_2$$

$$\left(\partial g / \partial X3\right)_f = -\,(70*x_1*x_4 + 17x_4*x_4 + 145\,x_4)\,\sigma_3$$

$$\left(\partial g / \partial X4\right)_f = -\,(70*x_1*x_3 + 34*x_3*x_4 + 145*x_3)\,\sigma_4$$

For first iteration, we should take expectations as the initial values.

$$\left(\partial g / \partial X1\right)_f = (3400-70*1*5.09)0.096=292.19,$$

$$\left(\partial g / \partial X2\right)_f = 0.64*170 = 108.80.$$

$$\left(\partial g / \partial X3\right)_f = -\,(70*0.64*5.09+17*5.09*5.09+145*5.09)0.02 = -283.97.$$

$$\left(\partial g / \partial X4\right)_f = -\,(70*0.64*1.0+34*1.0*5.09+145*1.0)0.889 = -322.67$$

Direction cosines ,$\boldsymbol{\alpha}_i$

$$\alpha_i = \frac{\left(\partial g / \partial Xi\right) f}{\sqrt[2]{\left(\sum \left(\partial g / \partial Xi\right) * \left(\partial g / \partial Xi\right)\right)}}$$

$$\alpha_1 = 292.19 \Big/ \sqrt{(2.8*10e5)} = 0.550$$

$$\alpha_2 = 108.80 \Big/ \sqrt{(2.8*10e5)} = 0.205$$

$$\alpha_3 = -283.97 \Big/ \sqrt{(2.8*10e5)} = -0.535$$

$$\alpha_4 = -322.67 \Big/ \sqrt{(2.8*10e5)} = -0.608$$

New failure point is given by

$x_{1(new)} = \mu_1 - \alpha_1 * \sigma_1 * \beta = 0.64 - 0.053\ \beta$ - (2)

$x_{2(new)} = \mu_2 - \alpha_2 * \sigma_2 * \beta = 3400 - 34.85\ \beta$ -(3)

$x_{3(new)} = \mu_3 - \alpha_3 * \sigma_3 * \beta = 1 + 0.107\ \beta$ - (4)

$x_{4(new)} = \mu_4 - \alpha_4 * \sigma_4 * \beta = 5.09 + 0.541\ \beta$ - (5)

By substituting (2), (3), (4), (5) in limit state equation (1), we get solution for reliability index, β =1.379.

Iteration process

	I iteration	II iteration	III iteration	IV iteration	V iteration
Initial x_{1f}	o.64	0.576	0.603	0.594	0.597
Initial x_{2f}	3400	3352	3378	3370	3373
Initial x_{3f}	1.00	1.147	1.088	1.105	1.009
Initial x_{4f}	5.16	5.825	5.637	5.704	5.681
$F(x^*_{4f})$	0.570	0.799	0.784	0.767	0.760

$F(x^*_{4f})$	4.4*10e-1	2.5*10e-1	3.0*10e-1	2.8*10e-1	2.9*10e-1
$inv\Phi[F(x^*_{4f})]$	0.177	0.838	0.668	0.729	0.708
$Ø\{inv\Phi[F(x^*_{4f})]\}$	0.393	0.281	0.319	0.306	0.311
Mean of X^*_{4f}	5.090	5.590	5.424	5.481	5.461
Standard deviation of X^*_{4f}	0.899	1.136	1.065	1.090	1.081
$\left(\partial g / \partial X1\right)_f$	292.19	278.69	284.59	282.86	283.47
$\left(\partial g / \partial X2\right)_f$	108.80	96.42	102.58	100.94	101.53
$\left(\partial g / \partial X3\right)_f$	-283.97	-312.40	-311.15	-314.95	-313.6
$\left(\partial g / \partial X4\right)_f$	-322.67	-488.34	-430.68	-449.30	-442.7
$\Sigma\left(\partial g / \partial Xi\right)f$	2.8*10e5	2.8*10e5	2.8*10e5	2.8*10e5	2.8*10e5
α_1	0.550	0.525	0.536	0.533	0.534
α_2	0.205	0.182	0.193	0.190	0.191
α_3	-0.535	-0.605	-0.586	-0.593	-0.590
α_4	-0.608	-0.920	-0.811	-0.846	-0.831
New x_{1f}	0.567	0.603	0.594	0.597	0.591
New x_{2f}	3352	3378	3370	3373	3372
Newx_{3f}	1.147	1.088	1.105	1.099	1.101
Newx_{4f}	5.836	6.348	6.201	6.525	6.234
g	4*10e-5	6.3*10e-5	-3*10e-5	-2.1*10e-5	-2.5*10e-5
β	1.379	0.726	0.899	0.837	0.83

Reliability $\Phi(\beta) = 0.805$

Risk $1- \Phi(\beta) = 0.195$