Example on PDF and CDF

The undrained shear strength c_u of a stratum of clay has a uniform probability distribution, the maximum and minimum values of uniform distribution being 25 kN/m^2 and 50 kN/m^2. What is the probability that the undrained shear strength has magnitude (a)less than 40 kN/m^2,(b)less than 30kN/m^2 (c)less than 10 kN/m^2 and (d) greater than 55 kN/m^2.

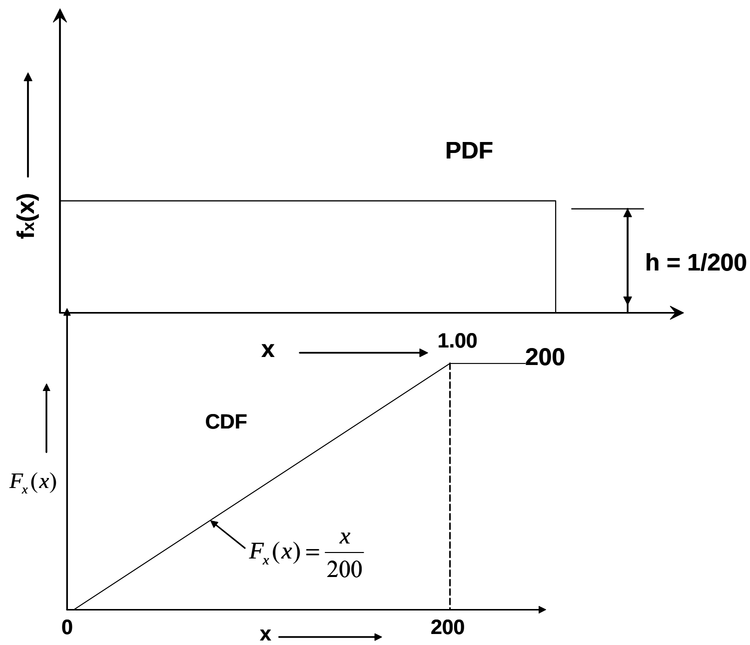


Figure 1 – Uniform probability density function with associated CDF

The area under the probability density function must be unity. In this case the abscissa or the rectangle is (50-25)=25. Therefore the height or the base of the rectangle (i.e. the uniform probability density p_x) is given by equating the area to 1.

$$p_{c_u} \times 25 = 1, \therefore\ p_{c_u} = \frac{1}{25}$$

We use this value as follows:

(a) $p_1 = P(c_u \leq 40)$

This probability is the area of the rectangle between the ordinates at c_u=25 (minimum value) and c_u=4O

$$\therefore\ p_1 = (40-25)\times\frac{1}{25} = 0.6$$

(b) $p_2 = P(c_u \le 30) = (30-25)\times\frac{1}{25} = 0.2$

(c) 10 kN/m^2 is outside the range 25—50, Accordingly $p_3 = P(c_u \le 10) = 0$

(d) 55 kN/m^2 is outside the range 25— 50. Accordingly, $p_4 = P(c_u > 55) = 0$

Example on Normal distribution

Example 1

From records, the total annual rainfall in a catchments area is estimated to be normal with a mean of 60 inches and standard deviation of 15 inches

a. What is the probability that in future years the annual rainfall will be between 40 to 70 inches
b. What is the probability that the annual rainfall will be at least 30 inches
c. What is the 10 percentile annual rainfall

a.

$$P(a \le X \le b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$S = \frac{x-\mu}{\sigma} \qquad dx = \sigma\, ds$$

$$P(a \le X \le b) = \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$$

$$P(40 \le X \le 70) = \phi\left(\frac{70-60}{15}\right) - \phi\left(\frac{40-60}{15}\right)$$

$$= \phi(0.67) - \phi(-1.33)$$

$$= \phi(0.67) - [1 - \phi(1.33)]$$

$$= 0.748571 - (1 - 0.908241) \qquad \textit{from table}$$

$$0.6568$$

b.

$$P(X \ge 30) = \phi(\infty) - \phi\left(\frac{30-60}{15}\right)$$

$$= 1 - \phi(-2.0)$$

$$= 1 - [1 - \phi(2.0)]$$

$$= 0.9772$$

$$P(X \ge 60) = \phi(\infty) - \phi(0) = 1.0$$

c.

$$P[X \le x_{0.1}] = 0.10$$

$$\phi\left(\frac{x_{0.1} - 60}{15}\right) = 0.10$$

$$\frac{x_{0.1} - 60}{15} = \phi^{-1}(0.10)$$

$$= -\phi^{-1}(0.90)$$

$$= -1.28$$

$$x_{0.10} = 60 - 19.2$$

$$= 40.8 \textit{ inches}$$

Example 2.

A structure is resting on three supports A,B and C. Even though the loads from the roof supports can be estimated accurately , this soil conditions under A, B and C are not completely predictable. Assume that the settlement ρ_A, ρ_B *and* ρ_C are independent , normal variates with mean of 2,2.5 and 3 cm and CoV of 20%, 20% and 25% respectively.

a. What is the probability that the maximum settlement will exceed 4cm
b. If I is known that A and B have settled 2.5cm and 3.5cm respectively . What is the probability that the maximum differential settlement will not exceed .8m and also what if it does not exceed 1.5 cm

Answer

a.

$$P(\max \rho > 4\,cm) = 1 - \rho(\max \rho \le 4\ cm)$$
$$SD = Mean * CoV$$
$$\sigma_A = 2 * 0.2 = 0.4$$
$$\sigma_B = 2.5 * 0.2 = 0.5$$
$$\sigma_C = 3 * 0.25 = 0.75$$

$$= 1 - P(\rho_A \le 4)P(\rho_B \le 4)P(\rho_C \le 4)$$
$$= 1 - \phi\left(\frac{4-2}{0.4}\right)\phi\left(\frac{4-2.5}{0.5}\right)\phi\left(\frac{4-3}{0.75}\right)$$
$$= 1 - \phi(5)\phi(3)\phi(1.333)$$
$$= 1 - 1 * 0.9986 * 0.9088$$
$$= 0.0925$$

$$\rho(\max\ \rho > 3\,cm) = 1 - P(\max \rho \leq 3)$$
$$= \rho(\max\ \rho > 3\,cm) = 1 - P(\ \rho_A \leq 3)P(\rho_B \leq 3)P(\rho_C \leq 3)$$
$$= 1 - \phi\left(\frac{3-2}{0.4}\right)\phi\left(\frac{3-2.5}{0.5}\right)\phi\left(\frac{3-3}{0.75}\right)$$
$$= 1 - 0.994 * 0.84 * 0.5$$
$$= 0.582$$

b.

The differential settlement between A and B is Δ_{AB} =3.5cm – 2.5cm = 1 cm has already occurred. Hence, $P(\Delta_{\max} \leq 0.8\ cm) = 0$ irrespective of ρ_C, however ρ_C matters if $P(\Delta_{\max} \leq 1.5\ cm)$, it is necessary if we need the data with 95% or 99% or 99.9% of occurrence , we need to determine the following

$$P(\mu - 1.960\sigma < X \leq \mu + 1.96\sigma) = 95\%$$
$$P(\mu - 2.58\sigma < X \leq \mu + 2.58\sigma) = 99\%$$
$$P(\mu - 3.29\sigma < X \leq \mu + 3.29\sigma) = 99.9\%$$

For the $\Delta_{\max}$ to be more than 1.5cm, $\rho_A = 2.5\ cm$, either ρ_C should be less than 1 or more than 4. Since $\rho_B = 3.5\ cm$, ρ_C should be less than 2cm or more than 5 cm . Acceptable region for safety is ρ_C should be between 2 to 4.

$$P(\Delta_{\max} \leq 1.5cm) = P(2\,cm \leq \rho_C \leq 4\,cm)$$
$$= \phi\left(\frac{2-3}{0.75}\right) - \phi\left(\frac{4-3}{0.75}\right)$$
$$= \phi(-1.333) - \phi(-1.333)$$
$$= 0.9088 - 0.0912$$
$$= 0.8176$$

Example 3

The total load on the footing is the sum of dead load of the structure and the live load. Since each load is sum of various components, total dead load (X) and total live load (Y)can be considered as normally distributed. The data from building suggest that $\mu_x = 100\ ton \quad and \quad \sigma_x = 10\,ton$ and $\mu_y = 40\ ton \quad and \quad \sigma_y = 10\,ton$ both x and y are not correlated. What is the total design load that has 5 % probability?

The total load Z = x + y = 100 + 40 = 140 ton and

$$\sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2} = \sqrt{10^2 + 10^2} = 10\sqrt{2} = 14.1\,ton$$

We need to determine z such that it has only 5% probability of occurrence

$$1 - \phi(\bar{z}) = 0.05$$
$$\phi(z) = 0.95$$
$$\phi\left(\frac{z-\mu}{\sigma}\right) = 0.95$$
$$\phi\left(\frac{z-140}{14.1}\right) = 0.95$$
$$\frac{z-140}{14.1} = 1.65$$
$$z = 163.3\ ton$$

Example 4

The mean and coefficient of variation of the angle of internal friction φ of a soil supporting a multi-storeyed structure are φ =20 and V φ =3O%. What is the probability that φ will be less than (a) 16^{o}, (b) 10^{o}, (c) 5^{o}, Assume that φ has a normal distribution

Solution

(a)

$$V_{\phi} = \frac{S_{\phi}}{\phi} = 0.3$$

$$S_{\phi} = 0.3 \times 20^{\circ} = 6^{\circ}$$

$$s = \frac{\phi - \overline{\phi}}{S_{\phi}} = \frac{16-20}{6} = \frac{-4}{6} = -0.666$$

$$\Phi(-s) = 1 - \Phi(-s) = 1 - \Phi(-0.666)$$

$$\textit{where } \Phi() \textit{ is obtained from tabulated values}$$

$$\therefore P(s \leq 16^{\circ}) = 0.253$$

(b)

$$s = \frac{10-20}{6} = -\frac{5}{3} = -1.666$$

$$\therefore \Phi(-1.666) = 1 - \Phi(1.666) = 1 - 0.952 = 0.048$$

$$P(s \leq 10^{\circ}) = 0.048 = 0.048 \times 10^{-1}$$

(c)

$$s = \frac{5-20}{6} = -\frac{15}{6} = -\frac{5}{2} = -2.5$$

$$\therefore \Phi(-2.5) = 1 - \Phi(2.5) = 1 - 0.994 = 0.006 = 6 \times 10^{-3}$$

It should be noted that φ denotes the friction angle and φ (s) is cumulative distribution of standard normal variate.

Example on cumulative distribution

Example 1

In the case of previous car problem

$0.00^6 + 0.04 + 0.12 + 0.21 + 0.25 = 0.620$ becomes the probability that 5 or less cars take a left turn.

If x is a random variable and r is a real number then the CDF designated as F(r) is the probability that x will take an value equal to or less than r or

$$F(r) = P[x \leq r]$$

For binomial distribution

$$F(r) = P[x \leq r] = \Sigma_{all\ x_t, C, r} b(x_i, N, R)$$

Though this distribution is quite simple, the model as such is quite useful in many engineering problems. For example in a series of soil boring, boulders may or may not be present.

Though the distribution is for discrete variables it can also be applied to continuous variables in space and time with discretisation.

Example on lognormal distribution

Example 1

The rainfall has lognormal distribution with mean and SD of 60 inches and 15 inches

a. Calculate the probability that in future the annual rainfall in between 40 and 70
b. The probability that the annual rainfall is at least 30”
c. What is the 10 percentile annual rainfall

$$\xi = \frac{15}{60} = 0.25$$

$$\lambda = \ln 60 - \frac{1}{2} * 0.25^2 = 4.09 - 0.03 = 4.06$$

a. The probability that the annual rainfall in between 40 and 70 is

$$P(40 < x \le 70) = \phi\left(\frac{\ln 70 - 4.06}{0.25} - \frac{\ln 40 - 4.06}{0.25}\right)$$
$$= \phi(0.75) - \phi(-1.48)$$
$$= 0.773 - 0.069$$
$$= 0.7039$$

b. The probability that annual rainfall is atleast 20 inches

$$P(x \ge 30) = 1 - \phi\left(\frac{\ln 30 - 4.06}{0.25}\right)$$
$$= 1 - \phi(2.64)$$
$$= 1 - 0.9959$$
$$= 0.041$$

c. 10 percentile rainfall

$$\phi\left(\frac{\ln x_{10} - 4.06}{0.25}\right) = 0.10$$
$$\frac{\ln x_{10} - 4.06}{0.25} = -1.28$$
$$\ln x_{0.10} = 4.06 - 1.28 - 0.25 = 3.74$$
$$x_{0.10} = e^{3.74} = 42.10$$

Example 2

A live load of 20ton acts on a structure of the loading is assumed to be log-normal distribution. Estimate the probability that a load of 40 will be exceeded. Assume CoV for live load = 25%

We have

$$\sigma[x] = \sqrt{\ln(CoV)^2}$$
$$\sigma[x] = \ln E(y) - (CoV)^2$$
$$\sigma[x] = \sqrt{\ln(1 + 0.25)^2} = 0.25$$
$$E[x] = \ln 20 - (0.25)^2 / 2$$

As x = lnL the value of the normal variate x equivalent 20 =ln 40 =3.69

Hence

$$h = \frac{3.69 - 2.96}{0.25} = 2.92$$

$$P[40 \leq L] = 0.5 - 4(2.92) = 0.5 - 0.498 = 0.002$$

Example on beta distribution

The ϕ of the soil samples in a locality varies between 20º to 40º with the coefficient of variation of 20º with mean value 30º. The design value is 35º. What is the probability that $\phi \geq 35^{o}$.

$$\mu_x = a + \frac{q}{(q + r)}(b - a)$$
$$30 = 20 + \frac{q}{(q + r)} * 20$$
$$\frac{q}{(q + r)} * 20 = 10$$
$$10q + 10r - 20q = 0$$
$$10r - 10q = 0$$
$$r = q$$

$$\sigma_x^2 = \frac{qr}{(q+r)^2(q+r+1)}(b-a)^2$$

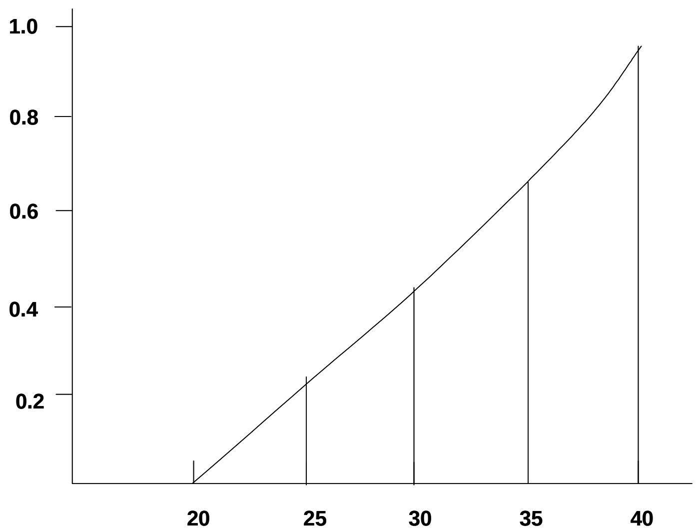
$$36 = \frac{qr}{(q+r)^2(q+r+1)}20^2$$

$$0.09 = \frac{qr}{(q+r)^2(q+r+1)} = \frac{q^2}{(q+q)^2(q+q+1)} = \frac{q^2}{4q^2 * (2q+1)}$$

$$1/4 = (2q+1) * 0.09$$

$$q = 5.05$$

$$r = 5.05$$



$$a + \frac{1-q}{2-q-r}(b-a)$$

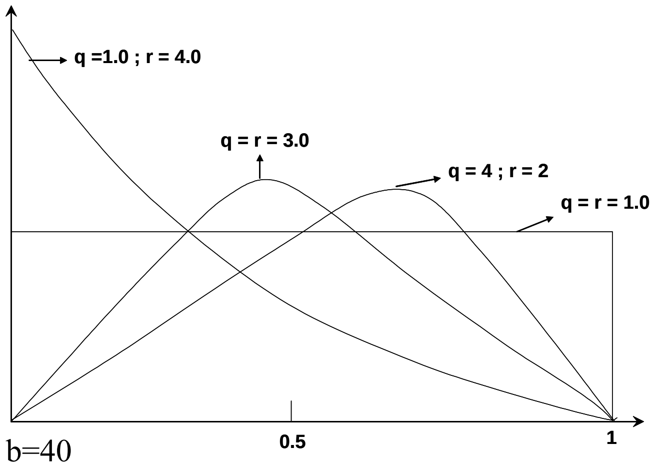
$$= 20 + \frac{1-5.05}{2-5.05*2} * 20$$

$$= 20 + \frac{-4.05}{-8.1} * 20$$

mode $\bar{x}$ $= = 30$

Coefficient of skew ness = $\dfrac{2(q-r)}{(q+r)(q+r+2)\sigma_x}$

q < r , the distribution is positive and skewed to the left q > r , the distribution is negative and skewed to the right



a=20 b=40

$$\mu_x = a + \frac{q}{(q+r)} + 20$$

$$\frac{6}{20} = \frac{q}{q+r} \Rightarrow 6q + 6r - 20q = 0$$

$$6r - 14q = 0$$

$$\sigma_x^2 = \frac{qr}{(q+r)(q+r+1)}(b-a)^2$$

$$SD = CoV * 26^{\circ}(5.2)^2$$

$$5.2^2 = \frac{qr}{(q+r)(q+r+1)}20^2$$

$$0.0676 = \frac{qr}{(q+r)(q+r+1)}$$

$$= \frac{\frac{6}{14}r * r}{\left(\frac{20r}{14}\right)*\left(\frac{20r+14}{14}\right)} = \frac{14*6r^2}{20r(20r+14)}$$

$$1.352 * r(20r+14) = 84r^2$$

$$27.04r^2 + 18.93r - 84r^2 = 0$$

$$-56.96r^2 = 18.93r$$

$$r = 0.33$$

$$q = 0.14$$

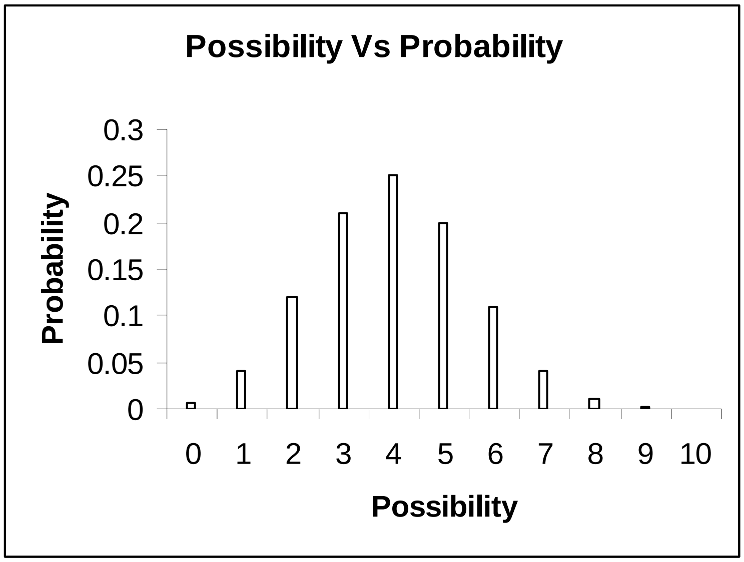
Example on binomial distribution

Example 1

Over a period of time it is observed that 40% of the automobiles traveling along a road will take a left turn at a given intersection. What is the probability that given a traffic stream of 10 automobiles 2 will take a left turn.

N= 10 ; x=2 ; R= 0.40

Possibility	Probability
0	0.006
1	0.04
2	0.12
3	0.21
4	0.25
5	0.20
6	0.11
7	0.04
8	0.01
9	0.002
10	0.0001



Example 2

A dam has a projected life of 50years . What is the probability that 100years flood will occur during the life time of the dam?

R = 1 / 100 = 0.01 , N = 50 years

$b(1.50,0.01) = 50\ (0.01)^{1}(0.99)^{49} = 0.31$

N		
10	$10*(0.01)^{1}*(0.99)^{9}$	0.09
20	$20*(0.01)^{1}*(0.99)^{19}$	0.165
30	$30*(0.01)^{1}*(0.99)^{29}$	0.224
40	$40*(0.01)^{1}*(0.99)^{39}$	0.270
50	$50*(0.01)^{1}*(0.99)^{49}$	0.305
60	$60*(0.01)^{1}*(0.99)^{59}$	0.387
70	$70*(0.01)^{1}*(0.99)^{69}$	0.350
80	$80*(0.01)^{1}*(0.99)^{79}$	0.362
90	$90*(0.01)^{1}*(0.99)^{89}$	0.368
100	$100*(0.01)^{1}*(0.99)^{99}$	0.370

Example 3

A flood control system for a river , the yearly maximum flood of river is concern. The probability of the annual maximum flood exceeding some specified design level no. is 0.1 in any year. What is the probability that the level no. will be exceeded once in five years.

Assuming binomial distribution means that there is only one occurrence or not in the year. Each occurrence or not occurrence is independent of the other events. The probability of occurrence in each trial is also considered.

$$b(x, N, R) = \binom{n}{x} p^x q^{(n-x)}$$

$$\frac{1!}{0!\ 1!} = \frac{n!}{(n-x)!\,x!} p^x q^{(n-x)}$$

$$= \frac{5!}{4!\ 1!} (0.1)^1 (0.9)^4$$

$$= 5 * (0.1)^1 (0.9)^4$$

Hence $= 0.328$

The probability that at least one exceedance of level no =

$P(x \le 1) = P(x = 0) + P(x = 1) = 0.590$

Example on binomial distribution

Example 1

1. Find the variance of the binomial distribution $b(x_i,N,R)$

$= b(x_i , 5 , 0.01)$

$$E[x/b(x_i,5,0.01)] = NR = 5*0.1 = 0.5$$
$$V[x_i] = E[(x_i - \bar{x})]^2$$
$$V[x_i] = E[x_i^2 - 2x_i\bar{x} + \bar{x}^2]$$
$$V[x_i] = E[x_i^2] - 2\bar{x}E[x_i] + E[\bar{x}^2]$$
$$V[x_i] = E[x_i^2] - 2\bar{x}^2 + \bar{x}^2$$
$$V[x_i] = E[x_i^2] - \bar{x}^2$$

Variance has the dimension of a square of the random variable ,more meaningful measure of dispersion is the positive square root of variance called standard deviation

$$\sigma[x_i] = \sqrt{\sqrt{[x_i]}}$$

The equivalent concept of standard deviation in static's is radius of gyration.

Another useful relative measure of scatter of radius of gyration called co-efficient of variation

$$V(x) = \frac{\sigma[x]}{E[x]} * 100\%$$

Coefficient of Variation express a measure of central tendency .For example a mean value of 10 and SD of 1 indicated 10% CoV .

10% < Low

15-30% = moderate

30% = High

For symmetrical distribution, all moment's of odd order are zero. The third central moment $E(x_i - \bar{x})^3$ represents the degree of skew ness .The fourth moment provides a measure of peaked ness (Kurtosis) of the distribution to make these non-dimensional ,They are divided by standard deviation raised to cube or fourth order respectively.

Co-efficient of skew ness = $\beta(1) = \frac{E[(x_i - \bar{x})^3]}{(\sigma[x_i])^3}$

β(1) is +ve ,the long tail is on the right side of the mean , β(2) is negative ,long tail is on the left side

Co-efficient of Kurtosis or peaked ness

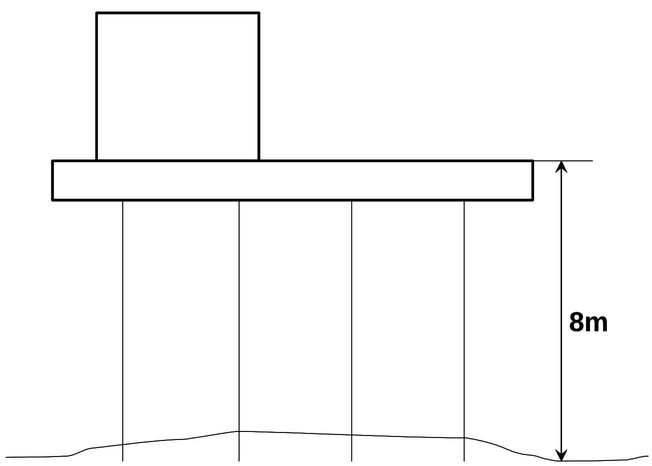
$$\beta(2) = \frac{E[(x_i - \bar{x})]^4}{\sigma[x]^4}$$

A distribution is said to be flat if $\beta(2) < 3$

Example on geometric distribution

Example 1

A structure is designed for a height 8m above the mean sea level. This height corresponds to 10 % probability of being exceeded by sea waves in a year. What is the probability that the structure will be subjected to waves exceeding 8m within return period of design wave.



$$T = \frac{1}{P} = \frac{1}{0.1} = 10 \ \textit{years}$$

$$P[H > 8M \ \ in\ 10 \ years] = 1 - (0.9)^{10} = 1 - 0.3487 = 0.6513$$

The number of trials (t) until specified event occurs for the first time is given by geometric distribution

$$b(1,1,p) = \frac{1!}{0!\ \ 1!} p^1 q^{(t-1)} = p * q^{(t-1)}$$

Example 2

A transmission tower is designed for 50years period i.e. a wind velocity having a return period of 50yrs

What is the probability that the design wind velocity will be exceeded for the first time in 5^{th} year, after the completion of the structure?

Every year is considered as scale and the probability of 50 years wind occurrence in any year is p=1 / 50 = 0.02

$$b(x, N, p, q) = p^x q^{(N-x)}$$
$$b(1,5,0.02,0.98) = (0.02)^1 (0.98)^4 = 0.0184$$

what is the probability that first such wind velocity will occur with 5 years after the completion of the structure.

$$P(T \le 5) = \sum_{i=1}^{5} (0.02)(0.98)^{t-1}$$
$$= 0.02 + 0.0196 + 0.0192 + 0.0188 + 0.0184$$
$$= 0.096$$

Example on Poisson's distribution

Example 1

Record of rain storm in a town indicates that on the average there have been four rainstorms per year over the last years. If the occurrence is assumed to follow a poisons process what is the probability that there is no rainstorm next year?

$$P(X_t = 0) = \frac{4^0}{0!} e^{-4} = 0.0018$$
$$= \frac{4^4}{4!} e^{-4} = 0.195$$

The above result indicate that the average yearly occurrence of rainstorm is 4, the probability of having exactly 4 storm in a year is also 4

The probability of 2 or more rainstorms in a year is

$$P(X_t \geq 2) = \sum_{x=2}^{\infty} \frac{4^x}{x!} e^{-4}$$
$$= 1 - \sum_{x=0}^{1} \frac{4^x}{x!} e^{-4} = 1 - 0.0018 - 0.024 = 0.908$$

Example 2

The probability that a structure fails is P(f) = 0.001 of 1000 such structures built what is the probability that two will fail

$$b(x, N, P(f)) = \frac{N!}{x!\ (N-x)!} P(f)^x R^{(N-x)}$$
Stirlings Formula
$$N! \approx \sqrt{2\pi N} N^N e^{-N}$$
$$\ln(N!) = \frac{1}{2}\ln(2\pi N) + N \ln N - N$$

$$b(x, N, P(f)) = \frac{1000!}{2!\ 998!} (0.001)^2 (0.999)^{998}$$
b
$$= \frac{998 * 1000}{2} = 0.184$$

to use Poisson distributions we need the expected value of binomial distribution

= n*p = 1000*0.001=1

$$f(2) = \frac{1^2 e^{-1}}{2!} = 0.184$$

$$f(0) = 0.37 \qquad f(1) = 0.37 \qquad f(2) = 0.18$$

$$f(3) = 0.06 \qquad f(4) = 0.02$$

Example 3

During world war II German dropped 54% bombs on London , an analysis was conducted to determine if the bombs were guided or not. It was reasoned that if bombs lacked guidance they should follow Poissons distribution. To check that, London was divided into 180 regions of approximately equal area and the number of hits in each area were recorded.

No of hits	Observed No. of areas within x_i	Poissons distribution with $f(x_i)$ with μ=0.943	Theoretical No.
0	229	0.389	226
1	211	0.367	213
2	93	0.173	100
3	39	0.054	32
4	7	0.013	7
5 or more	1	0.004	2
	580	1.00	580

The expected value per area = 547/580=0.943

From the above , one can say that bombing is a Poisson distribution

Example on exponential distribution

Example 1

Historical records of earthquake in San Francisco, California show that during the period 1836-1961 there were 16 earthquakes of intensity VI or more. If the occurrence of suc high intensity earthquakes in the region is assumed to follow a poisson process then what is the probability that such earthquakes will occur within the next years.

$$\gamma = \frac{16}{125} = 0.128 \ \ quakes \ \ per \ \ year$$
$$P(T_1 \le 2) = 1 - e^{-0.128(2)} = 0.226$$

The probability that no earthquake of this high intensity will occur in the next 10 years is

$$P(T_1 \ge 0) = e^{-10(0.128)} = 0.278$$
$$E(T_1) = \frac{1}{\gamma} = \frac{1}{0.128} = 7.8 \ \ years$$

Hence that an earthquake of at least VI intensity can be expected on an average once in very 7.8 years

Hence, the general model in the area is

$$P(T_1 \le t) = 1 - e^{-0.128t}$$
$$P(T_1 \le 7.8) = 1 - e^{-0.128*7.8} = 1 - e^{-1} = 0.632$$

Example on hyper geometric distribution

Example 1

A box contains 25 strain gauges and out of them 4 is known to be defective. If 6 gauges were used in the experiment, what is the probability that there is one defective gauge in the container.

N=25 , m=4 , n=6

The probability that one gauge is defective

$$P(X = 1) = \frac{\binom{4}{1}\binom{21}{5}}{\binom{25}{6}} = 0.46$$

The probability that no gauge is defective

$$P(X = 0) = \frac{\begin{matrix}21\\6\end{matrix}}{\begin{matrix}25\\6\end{matrix}} = 0.31$$

Example 2

An inspector on an highway project finds that two substandard test per 10 samples are a good measure of contractors ability to perform. Find the probability that I among the five samples selected at random

a. There is one substandard test
b. There are two such results

We have
$$P(X = x) = \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}$$

a. We have N = 10 , m = 2, x = 1, n = 5

$$= \frac{\binom{2}{1}\binom{8}{4}}{\binom{10}{5}} = \frac{\begin{matrix}21 & * & 81\\111 & & 4141\end{matrix}}{\begin{matrix}101\\5151\end{matrix}} = \frac{2*70}{252} = 0.55$$

b. We have N = 50, m = 5, n = 10, x = 1

$$= \frac{\binom{5}{1}\binom{45}{9}}{\binom{50}{10}} = \frac{\frac{5!}{4!1!} * \frac{45!}{36!9!}}{\frac{50!}{40!\ 10!}} = 0.43$$

Example 3

In an area chosen for foundation structure at 50 locations were collected and shear strength determined, If the 50,10 are unsuitable from shear strength considerations. In order to improve shear strength insitu grouting is one of the methods proposed for 10 locations. What is the probability that

a. the present location
b. Two locations were initially unsuitable

Answer

N = 50, m = 10, n = 10, x = 1 and 2

$$f(x) = \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}$$

a. The present location is unsuitable

$$f(1) = \frac{\binom{10}{1}\binom{50-10}{10-1}}{\binom{50}{10}} = \frac{\binom{10}{1}\binom{40}{9}}{\binom{50}{10}} = 0.267$$

$$f(2) = \frac{\binom{10}{2}\binom{50-10}{10-2}}{\binom{50}{10}} = \frac{\binom{10}{2}\binom{40}{8}}{\binom{50}{10}} = 0.34$$

if the location is such that, the one or two locations can be discarded leaving the ones in unsuitable area, the it is considered as a distribution with replacement. Binomial distribution can be chosen for the purpose

$$f(x) = \binom{n}{x}\left(\frac{M}{N}\right)^{x}\left(1-\frac{M}{N}\right)^{n-x}$$

$$f(1) = \binom{10}{1}\left(\frac{10}{50}\right)^{1}\left(1-\frac{10}{50}\right)^{9} = 0.268$$

$$f(2) = \binom{10}{2}\left(\frac{10}{50}\right)^{2}\left(1-\frac{10}{50}\right)^{9} = 0.30$$

Example 4

Records collected by a contractor over 40 years period indicate that there has been 240days of indigent weather, because of which the operations were closed down. On the basis of past record , what is the probability that no days were lost next year

Answer

Mean value of occurrence λ = 240 / 40 = 6 per year

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$f(0) = \frac{6^0 e^{-6}}{0!} = 2.48 * 10^{-3}$$

No of days	Probability of occurrence of 0.6
0	0.0025
1	0.0149
2	0.0446
3	0.0892
4	0.1339
5	0.1606
6	0.1606
7	0.1377
8	0.1033
9	0.0688
10	0.0413
11	0.0225
12	0.0113

The life period of materials / radioactivity are normally characterized in terms of exponential distribution.

$$f_T(t) = \lambda e^{-\lambda t}$$
$$f_T(0) = \lambda$$
$$f_T(\infty)$$

the corresponding distribution is

$$f_T(T) = \int_0^t \lambda e^{-\lambda t}$$

$$
\begin{aligned}
&= \lambda \left[-\frac{1}{\lambda} e^{-\lambda t} \right]_0^t \\
&= -\left[e^{-\lambda t} - 1 \right] = 1 - e^{-\lambda t}
\end{aligned}
$$

Mean or expected value

$$
\begin{aligned}
&E(k) = \int_{-\infty}^{\infty} x f_x \, dx \\
&Var(x) = \int_{-\infty}^{\infty} x f(x) dx \\
&hence \\
&\mu = E(T) = \int_{-\infty}^{\infty} t \lambda e^{-\lambda t} dt
\end{aligned}
$$

Example 5

A storm event occurring at a pint in a space is characterized by two variables, duration of the storm X and its intensity Y . if both the variables are exponentially distributed as

$$
\begin{aligned}
&F_x(x) = 1 - e^{-ax} \qquad x \geq 0 \quad a > 0 \\
&F_y(y) = 1 - e^{-by} \qquad y \geq 0 \quad b > 0
\end{aligned}
$$

Assuming that the joint distributions of both X and Y is also exponential then,

$$
\begin{aligned}
F_{X,Y} &= \left(1 - e^{-ax}\right)\left(1 - e^{-by}\right) \\
&= 1 - e^{-ax} - e^{-by} + e^{-(ax+by)} \\
\frac{\delta F_{X,Y}}{\delta x} &= +ae^{-ax} - ae^{-(+ax+ay)} \\
\frac{\delta^2 F_{X,Y}}{\delta x \delta y} &= abe^{-(ax+by)} \\
f_X(x) &= \int_0^{\infty} abe^{-(ax+by)} dy \\
&= \left[-\frac{ab}{b} e^{-(ax+by)} \right]_0^{\infty} \\
&= \left[-ae^{-(ax+by)} \right]_0^{\infty} \\
&= +ae^{-ax} \\
as \ y \to \infty \qquad & e^{-(ax+by)} \to 0 \\
f_y(y) &= be^{-by}
\end{aligned}
$$

$$f_x(x) = abe^{-(ax+by)}dxdy$$
$$= \int_0^\infty \int_0^\infty abe^{-(ax+by)}$$
$$= \int_0^\infty \left[\frac{abe^{-(ax+by)}}{-a} \right]$$
$$= \int_0^\infty -\left[0 - e^{-by}\right]dy$$
$$= \int_0^\infty e^{-by} dy$$
$$= \frac{1}{b}\left[e^{-by}\right]_0^\infty$$
$$= -\left[0. -1\right] = 1$$

The joint PDF for the concentration much of two pollutants (x,y) in ppms

If (x,y) = 2-x-y $0 \le x, y \le 1$

Show that

a. f(x,y) is a probability distributions
b. Determine CDF
c. The joint probability that $x \le 1/2$, $y \le 3/4$
d. Marginal distribution of x and y
e. Are they independent?
f. If the concentration of y is 0.5ppm, determine the probability $x \le 0.25$

Answer

a.

if f(x,y) in pd the volume has to be

$$\int_0^1\int_0^1 (2-x-y)dxdy = \int_0^1 \left|\left(2y - xy - \frac{y^2}{2}\right)\right|$$

$$= \int_0^1 |(2-x-1/2)|dx = \left[2x - x^2/2 - x/2\right]_0^1$$

$$= 2*1 - 1/2 - 1/2$$

$$= 1$$

b.

$$F_{(x,y)} = \int_0^x \int_0^y (2-4-0)$$

$$= \int_0^x \left[2y - \frac{y^2}{2} - 4y\right]dy$$

$$= \left[2xy - \frac{xy^2}{2} - xy\right]$$

c.

$$F(1/2, 3/4) = \left[2*1/2*3/4 - \frac{1/2*3^2/4^2}{2} - 1/2*3/4\right]$$

$$= 0.52$$

$$f_y(y) = \int_0^1 (2-x-y)dx = \left[2x - \frac{x^2}{2} - xy\right]_0^1 = \left[2 - \frac{1}{2} - y\right] = 1.5 - y$$

$$f_x(x) = \int_0^1 (2-x-y)dx = \left[2y - \frac{y^2}{2} - xy\right]_0^1 = \left[2 - \frac{1}{2} - x\right] = 1.5 - x$$

d.

$P(A \cap B) = P(A) * P(B)$

$$f_x(x) * f_y(y) = (1.5 - x)(1.5 - y) \; should \; be \; equal \; to \; f(xy)$$
$$= 2.25 - 1.5y - 1.5y$$
$$= 2 - x - y$$

similarly

$f_x(x) = ae^{-ax}$ $\qquad$ $f_y(y) = be^{-by}$

$f_x f_y = ab\, e^{-(ax+by)}$