Advanced Topics in Optimization

Multi Objective Optimization

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Introduction

Introduction

- In a real world problem it is very unlikely that we will meet the situation of single objective and multiple constraints more often than not
- There may be conflicting objectives along with the main objective like irrigation, hydropower, recreation etc.
- Generally multiple objectives or parameters have to be met before any acceptable solution can be obtained.
- Multi-criteria or multi-objective analysis are not designed to identify the best solution, but only to provide information on the tradeoffs.

Multi-objective Problem

- **A multi-objective optimization problem with inequality (or equality) constraints may be formulated as**

Find

$$X = \begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{Bmatrix} \qquad \textbf{(1)}$$

which minimizes $f_1(X), f_2(X), \ldots, f_k(X)$ **(2)**

subject to $g_j(X) \le 0,$ $j = 1, 2, \ldots, m$ **(3)**

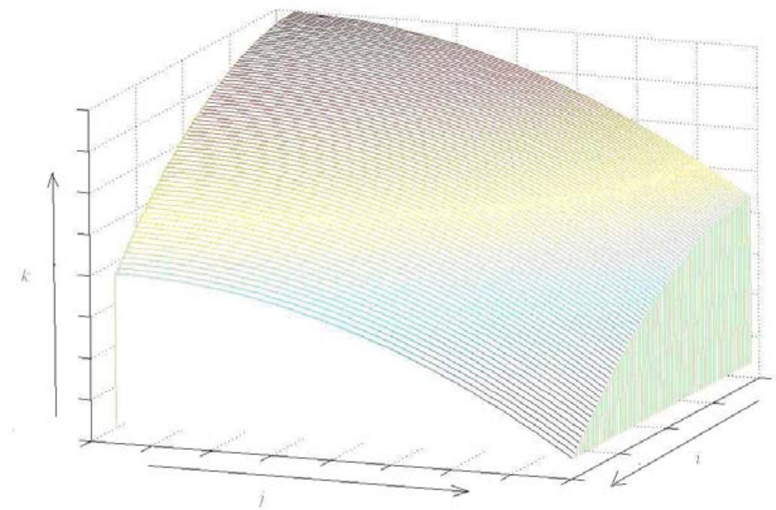
- ***Here k denotes the number of objective functions to be minimized and m is the number of constraints.***

Pareto Optimal Front

- A vector of the decision variable ***X*** is called Pareto Optimal (efficient) if there does not exist another ***Y*** such that $f_i(Y) \leq f_i(X)$ for *i* = 1, 2, … , *k* with $f_j(Y) < f_i(X)$ for at least one *j*
- In other words a solution vector ***X*** is called optimal if there is no other vector ***Y*** that reduces some objective functions without causing simultaneous increase in at least one other objective function.
- For the problems of the type mentioned above the very notion of optimization changes and we try to find good trade-offs rather than a single solution as in GP.

Pareto Optimal Front…

- As shown in above figure there are three objectives *i*, j, k. Direction of their increment is also indicated.
- The surface (which is formed based on constraints) is efficient because no objective can be reduced without a simultaneous increase in at least one of the other objectives.



Utility Function Method (Weighting function method)

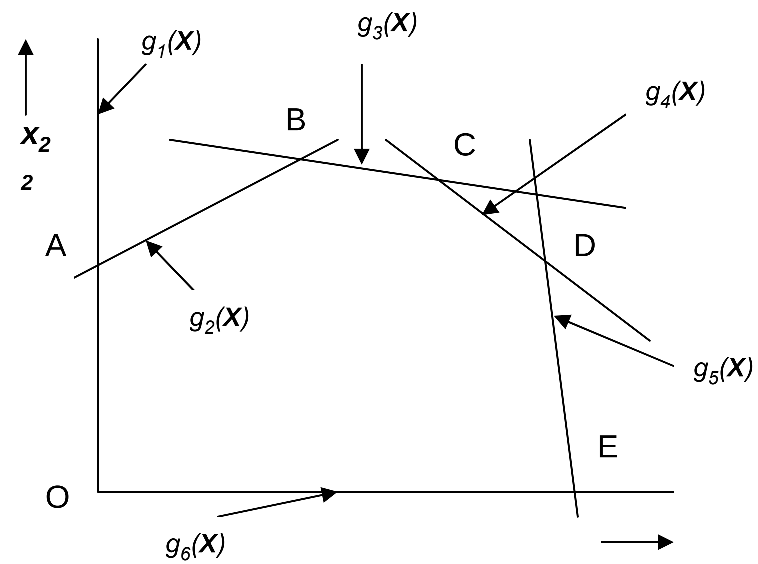
- In this method a utility function is defined for each of the objectives according to the relative importance of f_i.
- A simple utility function may be defined as $\alpha_i f_i(\mathbf{X})$ for ith objective where α_i is a scalar and represents the weight assigned to the corresponding objective.
- Total utility ***U*** may be defined as weighted sum of objective functions as below

$$\sum_{i=1}^{k} \alpha_i f_i(X), \qquad \alpha_i > 0, \; i = 1, 2, \ldots, k. \qquad (4)$$

- Without any loss in generality it is customary to assume that $\sum_{i=1}^{k} \alpha_i = 1$

Utility Function Method…

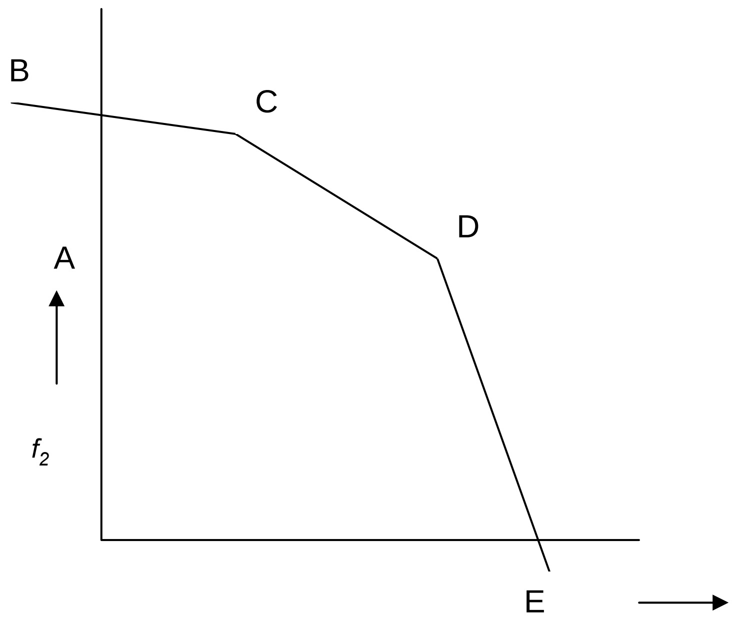
- Figure represents decision space for a given set of constraints and utility functions.
- Here $X =$ and two objectives are $f_1(X)$ and $f_2(X)$ with upper bound constraints* of type (3) as in figure 2.
- Pareto front is obtained by plotting the values of objective functions at common points (points of intersection) of constraints.



Decision Space

Utility Function Method…

- It should be noted that all the points on the constraint surface need not be efficient in Pareto sense as point A in the following figure.
- By looking at figure one may qualitatively infer that it follows Pareto Optimal definition.
- Now optimizing utility function means moving along efficient front and looking for the maximum value of utility function *U* defined by equation (4).



Bounded objective function method

- In this method we try to trap the optimal solution of the objective functions in a bounded or reduced feasible region.
- In formulating the problem one objective function is maximized while all other objectives are converted into constraints with lower bounds along with other constraints of the problem.
- Mathematically the problem may be formulated as

Maximize $f_i(\mathbf{X})$

Subject to

$$g_j(X) \leq 0, \quad j = 1, 2, \ldots, m \qquad (5)$$

$$f_k(\mathbf{X}) \geq e_k \qquad \forall \; k \neq i$$

here e_k represents lower bound on the kth objective.

Bounded objective function method…

- In this approach the feasible region **S** represented by $g_j(X) \le 0$, $j = 1, 2, \ldots, m$ is further reduced to **S'** by *(k-1)* constraints $f_k(\boldsymbol{X}) \ge e_k$.

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- e.g. let there are three objectives which are to be maximized in region of constraints S.
- Problem may be formulated

maximize{objective-1}
maximize{objective-2}
maximize{objective-3}

subject to

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S$$

- In above problem S identifies the region given by $g_j(X) \le 0$, $j = 1, 2, \ldots, m$.

Bounded objective function method…

- In bounded objective function method, same problem may be formulated as

  maximize{objective-1}

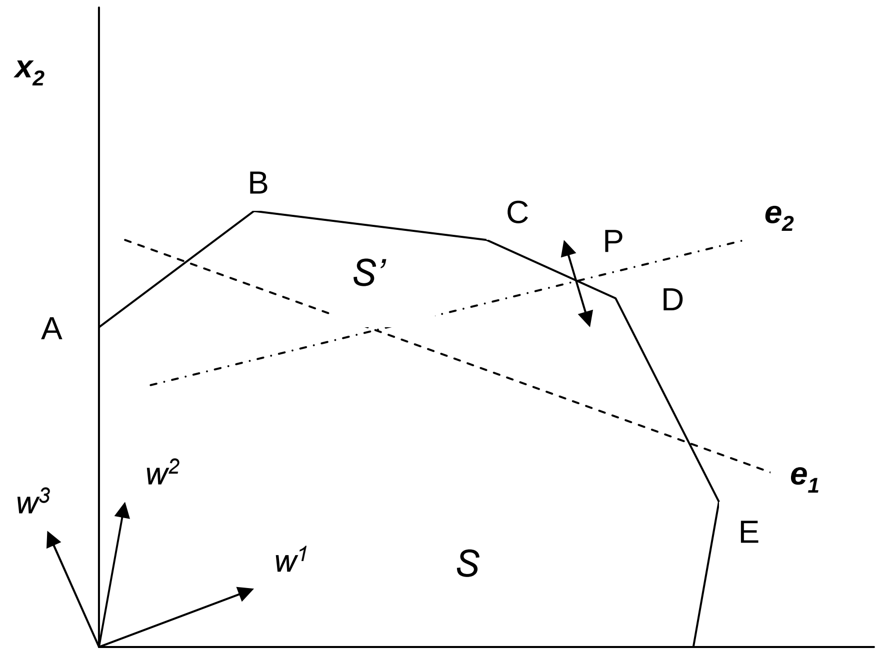
  subject to

  {objective-2} $\geq$ *e1*

  {objective-3} $\geq$ *e2*

  $\boldsymbol{X} \in$ **S**

- As may be seen that one of the objectives ({objective-1}) is now the only objective and all other objectives are included as the constraints.



Bounded objective function method…

- In previous figure w^1, w^2, and w^3 are gradients of the three objectives respectively.
- If {objective-1} was to be maximized in region *S* without taking into consideration other objectives then solution point had been *E*.
- But due to lower bound on other objectives the feasible region reduces to *S'* and solution point is P now.
- It may be seen that changing $\boldsymbol{e_1}$ does not affect {objective-1} as much as changing $\boldsymbol{e_2}$. This fact gives rise to sensitivity analysis.

Exercise Problem

- A reservoir is planned both for gravity and lift irrigation through withdrawals from its storage. The total storage available for both the uses is limited to 5 units each year. It is decided to limit the gravity irrigation withdrawals in a year to 4 units. If *X1* is the allocation of water to gravity irrigation and *X2* is the allocation for lift irrigation, two objectives are planned to be maximized and are expressed as

$$\text{Maximize} \quad Z_1(\mathbf{X}) = 3x_1 - 2x_2 \quad \text{and} \quad Z_2(\mathbf{X}) = -x_1 + 4x_2$$

For above problem do the following

(i) Generate a Pareto Front of non-inferior (efficient) solutions by plotting Decision space and Objective space.

(ii) Formulate multi objective optimization model using weighting approach with w_1 and w_2 as weights for gravity and lift irrigation respectively.

(iii) Solve it, if (a) w_1=1 and w_2=2 (b) w_1=2 and w_2=1

(iv) Formulate the problem using constraint method

[Solution: (i) $x_1=0$, $x_2=5$; (ii) $x_1=4$, $x_2=0$ *to* 1]

Thank You