Advanced Topics in Optimization

Piecewise Linear Approximation of a Nonlinear Function

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Introduction and Objectives

Introduction

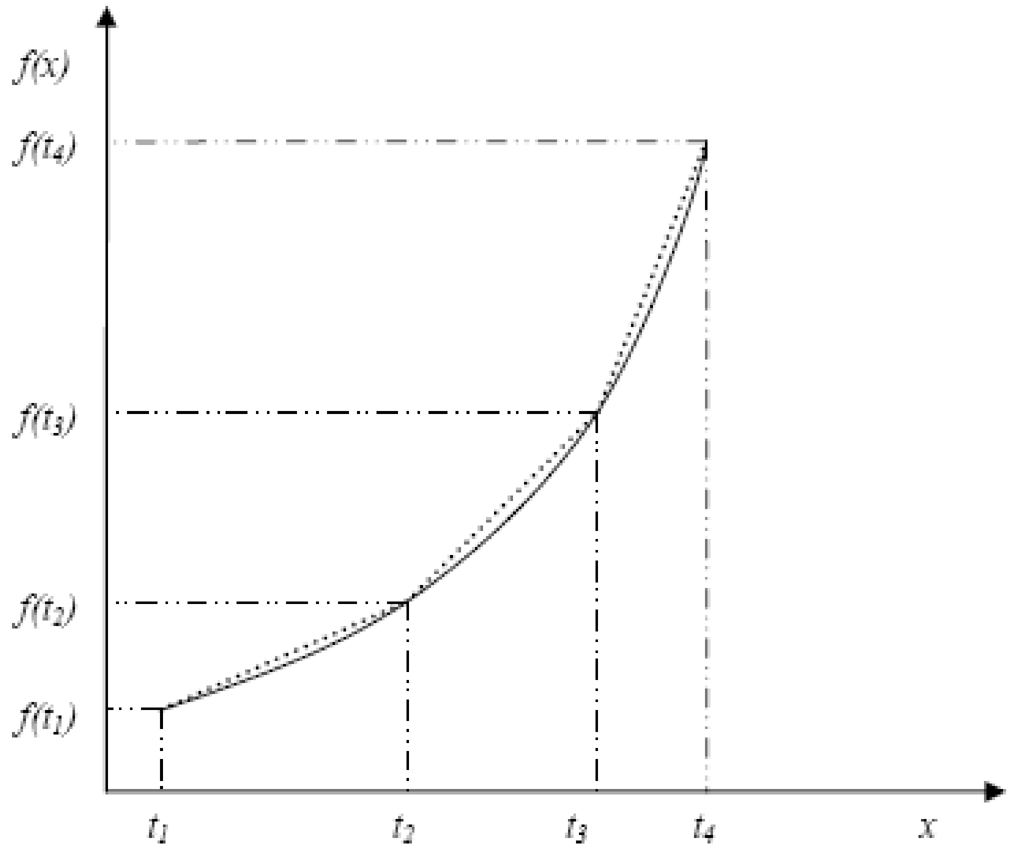
- There exists no general algorithm for nonlinear programming due to its irregular behavior
- Nonlinear problems can be solved by first representing the nonlinear function (both objective function and constraints) by a set of linear functions and then apply simplex method to solve this using some restrictions

Objectives

- To discuss the various methods to approximate a nonlinear function using linear functions
- To demonstrate this using a numerical example

Piecewise Linearization

- A nonlinear single variable function *f(x)* can be approximated by a piecewise linear function
- Geometrically, *f(x)* can be shown as a curve being represented as a set of connected line segments



Piecewise Linearization: Method 1

- Consider an optimization function having only one nonlinear term *f(x)*
- Let the x-axis of the nonlinear function *f(x)* be divided by *'p'* breaking points $t_1, t_2, t_2, \ldots, t_p$
- Corresponding function values be $f(t_1), f(t_2), \ldots, f(t_p)$
- If *'x'* can take values in the interval $0 \le x \le X$, then the breaking points can be shown as

$$0 \equiv t_1 < t_2 < \ldots < t_p \equiv X$$

Piecewise Linearization: Method 1 …contd.

- Express *'x'* as a weighted average of these breaking points

$$x = w_1 t_1 + w_2 t_2 + \ldots + w_p t_p$$

$$i.e., x = \sum_{i=1}^{p} w_i t_i$$

- Function *f(x)* can be expressed as

$$f(x) = w_1 f(t_1) + w_2 f(t_2) + \ldots + w_p f(t_p) = \sum_{i=1}^{p} w_i f(t_i)$$

where $\sum_{i=1}^{p} w_i = 1$

Piecewise Linearization: Method 1 …contd.

- Finally the model can be expressed as

$$Max \ \ or \ \ Min \ \ f(x) = \sum_{i=1}^{p} w_i f(t_i)$$

subject to the additional constraints

$$\sum_{i=1}^{p} w_i t_i = x$$

$$\sum_{i=1}^{p} w_i = 1$$

Piecewise Linearization: Method 1 …contd.

- This linearly approximated model can be solved using simplex method with some restrictions
- Restricted condition:
  - There should not be more than two 'w_i' in the basis and
  - Two 'w_i' can take positive values only if they are adjacent. i.e., if 'x' takes the value between t_i and t_{i+1}, then only w_i and w_{i+1} (contributing weights to the value of 'x') will be positive, rest all weights be zero
- In general, for an objective function consisting of 'n' variables ('n' terms) represented as

$$Max \;\; or \;\; Min \;\; f(x) = f_1(x_1) + f_2(x_2) + ... + f_n(x_n)$$

Piecewise Linearization: Method 1 …contd.

- subjected to *‘m’* constraints

$$g_{1j}(x_1)+g_{2j}(x_2)+....+g_{nj}(x_n)\le b_j \quad for\ j=1,2,...,m$$

- The linear approximation of this problem is

$$Max\ \ or\ \ Min\ \sum_{k=1}^{n}\sum_{i=1}^{p} w_{ki}\, f_k(t_{ki})$$

$$subjected\ \ to\ \sum_{k=1}^{n}\sum_{i=1}^{p} w_{ki}\, g_{kj}(t_{ki}) \le b_j \quad for\ j=1,2,...,m$$

$$\sum_{i=1}^{p} w_{ki} = 1 \qquad\qquad for\ k=1,2,...,n$$

Piecewise Linearization: Method 2

- 'x' is expressed as a sum, instead of expressing as the weighted sum of the breaking points as in the previous method

$$x = t_1 + u_1 + u_2 + ... + u_{p-1} = t_1 + \sum_{i=1}^{p-1} u_i$$

where u_i is the increment of the variable 'x' in the interval (t_i, t_{i+1}) i.e., the bound of u_i is $0 \le u_i \le t_{i+1} - t_i$

- The function *f(x)* can be expressed as

$$f(x) = f(t_1) + \sum_{i=1}^{p-1} \alpha_i u_i$$

- where α_i represents the slope of the linear approximation between the points t_{i+1} and t_i $\alpha_i = \frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i}$

Piecewise Linearization: Method 2 …contd.

- Finally the model can be expressed as

$$Max \;\; or \;\; Min \;\; f(x) = f(t_1) + \sum_{i=1}^{p-1} \alpha_i u_i$$

subjected to additional constraints

$$t_1 + \sum_{i=1}^{p-1} u_i = x$$

$$0 \leq u_i \leq t_{i+1} - t_i, \quad i = 1, 2, \ldots., p-1$$

Piecewise Linearization: Numerical Example

- The example below illustrates the application of method 1
- Consider the objective function

$$Maximize \quad f = x_1^3 + x_2$$

- subject to

$$2x_1^2 + 2x_2 \leq 15$$

$$0 \leq x_1 \leq 4$$

$$x_2 \geq 0$$

- The problem is already in separable form (i.e., each term consists of only one variable).

Piecewise Linearization: Numerical Example …contd.

- Split up the objective function and constraint into two parts

$$f = f_1(x_1) + f_2(x_2)$$
$$g_1 = g_{11}(x_1) + g_{12}(x_2)$$

where

$$f_1(x_1) = x_1^3;\; f_2(x_2) = x_2$$
$$g_{11}(x_1) = 2x_1^2;\; g_{12}(x_2) = 2x_2$$

- $f_2(x_2)$ and $g_{12}(x_2)$ are treated as linear variables as they are in linear form

Piecewise Linearization: Numerical Example …contd.

- Consider five breaking points for x_1

i	t_{1i}	$f_i(t_{1i})$	$g_{1i}(t_{1i})$
1	0	0	0
2	1	1	2
3	2	8	8
4	3	27	18
5	4	64	32

- $f_1(x_1)$ can be written as,

$$f_1(x_1) = \sum_{i=1}^{5} w_{1i} f_1(t_{1i})$$

$$= w_{11} \times 0 + w_{12} \times 1 + w_{13} \times 8 + w_{14} \times 27 + w_{15} \times 64$$

Piecewise Linearization: Numerical Example …contd.

- $g_{11}(x_1)$ can be written as,

$$g_{11}(x_1) = \sum_{i=1}^{5} w_{1i} g_{1i}(t_{1i})$$

$$= w_{11} \times 0 + w_{12} \times 2 + w_{13} \times 8 + w_{14} \times 18 + w_{15} \times 32$$

- Thus, the linear approximation of the above problem becomes

$$Maximize \quad f = w_{12} + 8w_{13} + 27w_{14} + 64w_{15} + x_2$$

subject to

$$2w_{12} + 8w_{13} + 18w_{14} + 32w_{15} + 2x_2 + s_1 = 15$$

$$w_{11} + w_{12} + w_{13} + w_{14} + w_{15} = 1$$

$$w_{1i} \geq 0 \; for \; i = 1,2,\ldots,5$$

Piecewise Linearization: Numerical Example …contd.

- This can be solved using simplex method in a restricted basis condition
- The simplex tableau is shown below

Iteration	Basis	f	Variables							b_r	$\frac{b_r}{c_{rs}}$
			w_{11}	w_{12}	w_{13}	w_{14}	w_{15}	x_2	s_1		
1	f	1	0	-1	-8	-27	-64	-1	0	0	--
	s_1	0	0	2	8	18	32	2	1	15	1.87
	w_{11}	0	1	1	1	1	1	0	0	1	1

Piecewise Linearization: Numerical Example …contd.

- From the table, it is clear that w_{15} should be the entering variable
- S_1 should be the exiting variable
- But according to restricted basis condition w_{15} and w_{11} cannot occur together in basis as they are not adjacent
- Therefore, consider the next best entering variable w_{14}
- This also is not possible, since S_1 should be exited and w_{14} and w_{11} cannot occur together
- The next best variable w_{13}, it is clear that w_{11} should be the exiting variable

Piecewise Linearization: Numerical Example …contd.

- The simplex tableau is shown below

Iteration	Basis	f	Variables							b_r	$\frac{b_r}{c_{rs}}$
			w_{11}	w_{12}	w_{13}	w_{14}	w_{15}	x_2	s_1		
1	f	1	8	7	0	-19	-56	1	0	8	--
	s_1	0	-8	-6	0	10	24	2	1	7	3
	w_{13}	0	1	1	1	1	1	0	0	1	15

- The entering variable is w_{15}. Then the variable to be exited is S_1 and this is not acceptable since w_{15} is not an adjacent point to w_{13}
- Next variable w_{14} can be admitted by dropping s_1.

Piecewise Linearization: Numerical Example …contd.

- The simplex tableau is shown below

Iteration	Basis	f	Variables							b_r	$\frac{b_r}{c_{rs}}$
			w_{11}	w_{12}	w_{13}	w_{14}	w_{15}	x_2	s_1		
1	f	1	-7.2	-4.4	0	0	-10.4	4.8	1.9	21.3	--
	w_{14}	0	-0.8	-0.6	0	1	2.4	0.2	0.1	0.7	
	w_{13}	0	1.8	1.6	1	0	-1.4	-0.2	-0.1	0.3	

- Now, w_{15} cannot be admitted since w_{14} cannot be dropped
- Similarly w_{11} and w_{12} cannot be entered as w_{13} cannot be dropped

Piecewise Linearization: Numerical Example …contd.

- Since there is no more variable to be entered, the process ends
- Therefore, the best solution is

$$w_{13} = 0.3;\ w_{14} = 0.7$$

- Now, $$x_1 = \sum_{i=1}^{5} w_{1i} t_{1i} = w_{13} \times 2 + w_{14} \times 3 = 2.7$$

$$and\ x_2 = 0$$

- The optimum value is $f = 21.3$
- This may be an approximate solution to the original nonlinear problem
- However, the solution can be improved by taking finer breaking points

Thank You