



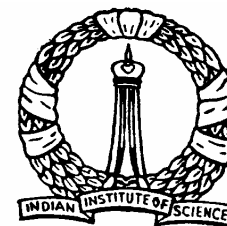
Integer Programming

Examples



Objectives

- To illustrate Gomory Cutting Plane Method for solving
 - All Integer linear Programming (AILP)
 - Mixed Integer Linear Programming (MILP)



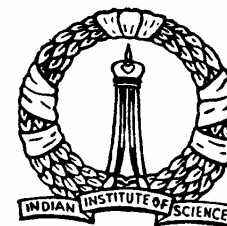
Example Problem (AILP)

Consider the problem

Maximize	$Z = 3x_1 + x_2$
subject to	$2x_1 - x_2 \leq 6$
	$3x_1 + 9x_2 \leq 45$
	$x_1, x_2 \geq 0$
	x_1, x_2 are integers

Standard form of the problem can be written as

Maximize	$Z = 3x_1 + x_2$
subject to	$2x_1 - x_2 + y_1 = 6$
	$3x_1 + 9x_2 + y_2 = 45$
	$x_1, x_2, y_1, y_2 \geq 0$
	x_1, x_2, y_1 and y_2 are integers



Example Problem (AIP) ...contd.

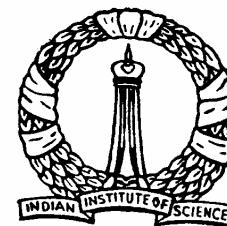
Solve the problem as an ordinary LP (neglecting integer requirements)

The final tableau of LP problem is shown below

Iteration	Basis	Z	Variables				b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	y_1	y_2		
3	Z	1	0	0	$\frac{8}{7}$	$\frac{5}{21}$	$\frac{123}{7}$	--
	x_1	0	1	0	$\frac{6}{14}$	$\frac{1}{21}$	$\frac{33}{7}$	--
	x_2	0	0	1	$-\frac{1}{7}$	$\frac{2}{21}$	$\frac{24}{7}$	--

Optimum value of Z is $\frac{123}{7}$ and the values of basic variables are

$$x_1 = \frac{33}{7} = 4\frac{5}{7}; \quad x_2 = \frac{24}{7} = 3\frac{3}{7}$$



Example Problem (AIP) ...contd.

Since the values of basic variables are not integers, generate Gomory constraint for x_1 which has a high fractional value. For this, write the equation for x_1 from the table above

$$x_1 = \frac{33}{7} - \frac{6}{14}y_1 - \frac{1}{21}y_2$$

Here, $b_1 = \frac{33}{7}$, $\bar{b}_i = 4$, $\beta_i = \frac{5}{7}$,

$$c_{11} = \frac{6}{14}, \bar{c}_{11} = 0, \alpha_{11} = \frac{6}{14},$$

$$c_{12} = \frac{1}{21}, \bar{c}_{12} = 0 \text{ and } \alpha_{12} = \frac{1}{21}$$

Thus, Gomory constraint can be written as

$$s_1 - \alpha_{11}y_1 - \alpha_{12}y_2 = -\beta_1$$

$$\text{i.e., } s_1 - \frac{6}{14}y_1 - \frac{1}{21}y_2 = -\frac{1}{21}$$

D Nagesh Kumar, IISc

Optimization Methods: M7L3

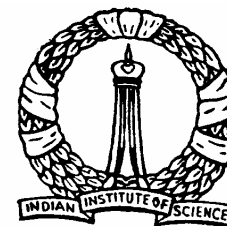


Example Problem (AIP) ...contd.

Insert this constraint as a new row in the previous table

Iteration	Basis	Z	Variables					b_r
			x_1	x_2	y_1	y_2	s_1	
	Z	1	0	0	$\frac{8}{7}$	$\frac{5}{21}$	0	$\frac{123}{7}$
	x_1	0	1	0	$\frac{6}{14}$	$\frac{1}{21}$	0	$\frac{33}{7}$
	x_2	0	0	1	$-\frac{1}{7}$	$\frac{2}{21}$	0	$\frac{24}{7}$
	s_1	0	0	0	$-\frac{6}{14}$	$-\frac{1}{21}$	1	$-\frac{5}{7}$

Solve this using Dual Simplex method

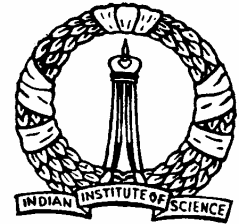


Example Problem (AIP) ...contd.

Iteration	Basis	Z	Variables					b_r
			x_1	x_2	y_1	y_2	s_1	
	Z	1	0	0	0	$\frac{1}{9}$	$\frac{8}{3}$	$\frac{47}{3}$
	x_1	0	1	0	0	0	1	4
4	x_2	0	0	1	0	$\frac{1}{9}$	$-\frac{1}{3}$	$\frac{11}{3}$
	y_1	0	0	0	1	$\frac{1}{9}$	$-\frac{7}{3}$	$\frac{5}{3}$

Optimum value of Z is $\frac{47}{3}$ and the values of basic variables are

$$x_1 = 4; \quad x_2 = \frac{11}{3}; \quad \text{and} \quad y_1 = -\frac{7}{3}$$



Example Problem (AIP) ...contd.

Since the values of basic variable x_2 from this table is not an integer, generate Gomory constraint for x_2 . For this, write the equation for x_2 from the table above

$$x_2 = 11\frac{1}{3} - \frac{1}{9}y_2 + \frac{1}{3}s_1$$

Thus, Gomory constraint can be written as

$$s_2 - \frac{1}{9}y_2 + \frac{1}{3}s_1 = -\frac{2}{3}$$

Insert this constraint as a new row in the last table and solve using dual simplex method

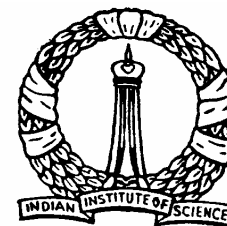


Example Problem (AIP) ...contd.

Iteration	Basis	Z	Variables						b_r
			x_1	x_2	y_1	y_2	s_1	s_2	
	Z	1	0	0	0	0	$\frac{8}{3}$	$\frac{1}{3}$	15
	x_1	0	1	0	0	0	1	0	4
5	x_2	0	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	3
	y_1	0	0	0	1	0	$-\frac{7}{3}$	$\frac{1}{3}$	1
	s_2	0	0	0	0	1	0	-3	6

Optimum value of Z is 15 and the values of basic variables are $x_1 = 4$, $x_2 = 3$, $y_1 = 1$, $s_2 = 6$ and $y_2 = s_1 = 0$.

These are satisfying the constraints and hence the desired solution.



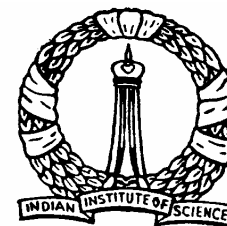
Example Problem (MILP)

Consider the previous problem with integer constraint only on x_2

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + x_2 \\ \text{subject to} \quad & 2x_1 - x_2 \leq 6 \\ & 3x_1 + 9x_2 \leq 45 \\ & x_1, x_2 \geq 0 ; \quad x_2 \text{ is an integer} \end{aligned}$$

Standard form of the problem can be written as

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + x_2 \\ \text{subject to} \quad & 2x_1 - x_2 + y_1 = 6 \\ & 3x_1 + 9x_2 + y_2 = 45 \\ & x_1, x_2, y_1, y_2 \geq 0 ; \quad x_2 \text{ should be an integer} \end{aligned}$$



Example Problem (MILP) ...contd.

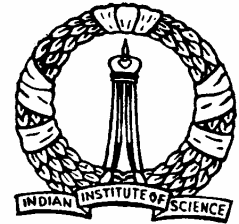
Solve the problem as an ordinary LP (neglecting integer requirements)

The final tableau of LP problem is shown below

Iteration	Basis	Z	Variables				b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	y_1	y_2		
3	Z	1	0	0	$\frac{8}{7}$	$\frac{5}{21}$	$\frac{123}{7}$	--
	x_1	0	1	0	$\frac{6}{14}$	$\frac{1}{21}$	$\frac{33}{7}$	--
	x_2	0	0	1	$-\frac{1}{7}$	$\frac{2}{21}$	$\frac{24}{7}$	--

Optimum value of Z is $\frac{123}{7}$ and the values of basic variables are

$$x_1 = \frac{33}{7} = 4\frac{5}{7}; \quad x_2 = \frac{24}{7} = 3\frac{3}{7}$$



Example Problem (MILP) ...contd.

Since the value of x_2 is not an integer, generate Gomory constraint for x_2 . For this, write the equation for x_2 from the table above

$$x_2 = 24/7 + 1/7 y_2 - 2/21 y_1$$

Here, $b_2 = 24/7$, $c_{21} = 1/7$, $c_{22} = -2/21$

Thus, the value of $\bar{b}_2 = 3$ and $\beta_2 = 3/7$

Since, $c_{21} = \bar{c}_{21}^+ + \bar{c}_{21}^-$ and $c_{22} = \bar{c}_{22}^+ + \bar{c}_{22}^-$,

$$\bar{c}_{21}^+ = 0, \bar{c}_{21}^- = -1/7 \quad \text{since } \bar{c}_{21} \text{ is negative}$$

$$\bar{c}_{22}^+ = 2/21, \bar{c}_{22}^- = 0 \quad \text{since } \bar{c}_{22} \text{ is positive}$$



Example Problem (MILP) ...contd.

Thus, Gomory constraint can be written as

$$s_i - \sum_{j=1}^m \bar{c}_{ij}^+ y_j - \frac{\beta_i}{\beta_i - 1} \sum_{j=1}^m \bar{c}_{ij}^- y_j = -\beta_i$$

$$\text{i.e., } s_2 - \frac{2}{21} y_2 - \frac{3}{28} y_1 = -\frac{3}{7}$$

Insert this constraint as a new row to the previous table and solve it using Dual Simplex method

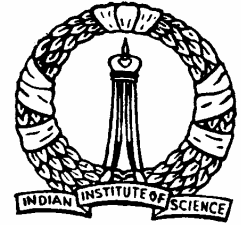


Example Problem (MILP) ...contd.

Iteration	Basis	Z	Variables					b_r
			x_1	x_2	y_1	y_2	s_2	
4	Z	1	0	0	$\frac{7}{8}$	0	1	$\frac{33}{2}$
	x_1	0	1	0	$\frac{3}{8}$	0	1	$\frac{9}{2}$
	x_2	0	0	1	$-\frac{1}{4}$	0	1	3
	y_2	0	0	0	$\frac{9}{8}$	1	$-\frac{21}{2}$	$\frac{9}{2}$

Optimum value of Z is $\frac{33}{2}$ and the values of basic variables are $x_1 = 4.5$; $x_2 = 3$; $y_2 = 4.5$ and that of non-basic variables are zero.

This solution is satisfying all the constraints and hence the desired.



Thank You