Integer Programming

Examples

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Objectives

- To illustrate Gomory Cutting Plane Method for solving
  - All Integer linear Programming (AILP)
  - Mixed Integer Linear Programming (MILP)

Example Problem (AILP)

Consider the problem

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + x_2 \\ \text{subject to} \quad & 2x_1 - x_2 \le 6 \\ & 3x_1 + 9x_2 \le 45 \\ & x_1, x_2 \ge 0 \end{aligned}$$

x_1, x_2 are integers

Standard form of the problem can be written as

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + x_2 \\ \text{subject to} \quad & 2x_1 - x_2 + y_1 = 6 \\ & 3x_1 + 9x_2 + y_2 = 45 \\ & x_1, x_2, y_1, y_2 \ge 0 \end{aligned}$$

x_1, x_2, y_1 and y_2 are integers

Example Problem (AILP) …contd.

Solve the problem as an ordinary LP (neglecting integer requirements)

The final tableau of LP problem is shown below

Iteration	Basis	Z	Variables				b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	y_1	y_2		
3	Z	1	0	0	$\frac{8}{7}$	$\frac{5}{21}$	$\frac{123}{7}$	--
	x_1	0	1	0	$\frac{6}{14}$	$\frac{1}{21}$	$\frac{33}{7}$	--
	x_2	0	0	1	$-\frac{1}{7}$	$\frac{2}{21}$	$\frac{24}{7}$	--

Optimum value of Z is $\frac{123}{7}$ and the values of basic variables are

$$x_1 = \frac{33}{7} = 4\,{}^5\!/_7; \quad x_2 = \frac{24}{7} = 3\,{}^3\!/_7$$

Example Problem (AILP) …contd.

Since the values of basic variables are not integers, generate Gomory constraint for x_1 which has a high fractional value. For this, write the equation for x_1 from the table above

$$x_1 = 33/7 - 6/14\, y_1 - 1/21\, y_2$$

Here, $b_1 = 33/7,\ \bar{b}_i = 4,\ \beta_i = 5/7,$

$$c_{11} = 6/14, \bar{c}_{11} = 0, \alpha_{11} = 6/14,$$

$$c_{12} = 1/21, \bar{c}_{12} = 0 \ and\ \alpha_{12} = 1/21$$

Thus, Gomory constraint can be written as

$$s_1 - \alpha_{11} y_1 - \alpha_{12} y_2 = -\beta_1$$

$$i.e.,\ s_1 - 6/14\, y_1 - 1/21\, y_2 = -1/21$$

Example Problem (AILP) …contd.

Insert this constraint as a new row in the previous table

Iteration	Basis	Z	Variables					b_r
			x_1	x_2	y_1	y_2	s_1	
	Z	1	0	0	$\frac{8}{7}$	$\frac{5}{21}$	0	$\frac{123}{7}$
	x_1	0	1	0	$\frac{6}{14}$	$\frac{1}{21}$	0	$\frac{33}{7}$
	x_2	0	0	1	$-\frac{1}{7}$	$\frac{2}{21}$	0	$\frac{24}{7}$
	s_1	0	0	0	$-\frac{6}{14}$	$-\frac{1}{21}$	1	$-\frac{5}{7}$

Solve this using Dual Simplex method

Example Problem (AILP) …contd.

Iteration	Basis	Z	Variables					b_r
			x_1	x_2	y_1	y_2	s_1	
4	Z	1	0	0	0	$\frac{1}{9}$	$\frac{8}{3}$	$\frac{47}{3}$
	x_1	0	1	0	0	0	1	4
	x_2	0	0	1	0	$\frac{1}{9}$	$-\frac{1}{3}$	$\frac{11}{3}$
	y_1	0	0	0	1	$\frac{1}{9}$	$-\frac{7}{3}$	$\frac{5}{3}$

Optimum value of Z is $\frac{47}{3}$ and the values of basic variables are

$x_1 = 4;\ x_2 = \frac{11}{3};$ and $y_1 = -\frac{7}{3}$

Example Problem (AILP) …contd.

Since the values of basic variable x_2 from this table is not an integer, generate Gomory constraint for x_2. For this, write the equation for x_2 from the table above

$$x_2 = 11/3 - 1/9\, y_2 + 1/3\, s_1$$

Thus, Gomory constraint can be written as

$$s_2 - 1/9\, y_2 + 1/3\, s_1 = -2/3$$

Insert this constraint as a new row in the last table and solve using dual simplex method

Example Problem (AILP) …contd.

Iteration	Basis	Z	Variables x_1	x_2	y_1	y_2	s_1	s_2	b_r
5	Z	1	0	0	0	0	$\frac{8}{3}$	$\frac{1}{3}$	15
	x_1	0	1	0	0	0	1	0	4
	x_2	0	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	3
	y_1	0	0	0	1	0	$-\frac{7}{3}$	$\frac{1}{3}$	1
	s_2	0	0	0	0	1	0	-3	6

Optimum value of Z is *15* and the values of basic variables are

$x_1 = 4$, $x_2 = 3$, $y_1 = 1$, $s_2 = 6$ *and* $y_2 = s_1 = 0$.

These are satisfying the constraints and hence the desired solution.

Example Problem (MILP)

Consider the previous problem with integer constraint only on x_2

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + x_2 \\ \text{subject to} \quad & 2x_1 - x_2 \le 6 \\ & 3x_1 + 9x_2 \le 45 \\ & x_1, x_2 \ge 0 \;; \quad x_2 \textit{ is an integer} \end{aligned}$$

Standard form of the problem can be written as

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + x_2 \\ \text{subject to} \quad & 2x_1 - x_2 + y_1 = 6 \\ & 3x_1 + 9x_2 + y_2 = 45 \\ & x_1, x_2, y_1, y_2 \ge 0 \;; \quad x_2 \textit{ should be an integer} \end{aligned}$$

Example Problem (MILP) …contd.

Solve the problem as an ordinary LP (neglecting integer requirements)

The final tableau of LP problem is shown below

Iteration	Basis	Z	Variables				b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	y_1	y_2		
3	Z	1	0	0	$\frac{8}{7}$	$\frac{5}{21}$	$\frac{123}{7}$	--
	x_1	0	1	0	$\frac{6}{14}$	$\frac{1}{21}$	$\frac{33}{7}$	--
	x_2	0	0	1	$-\frac{1}{7}$	$\frac{2}{21}$	$\frac{24}{7}$	--

Optimum value of Z is $\frac{123}{7}$ and the values of basic variables are

$$x_1 = \frac{33}{7} = 4\,{}^5\!/_7; \qquad x_2 = \frac{24}{7} = 3\,{}^3\!/_7$$

Example Problem (MILP) …contd.

Since the value of x_2 is not an integer, generate Gomory constraint for x_2. For this, write the equation for x_2 from the table above

$$x_2 = 24/7 + 1/7\, y_2 - 2/21\, y_1$$

Here, $\quad b_2 = 24/7,\ c_{21} = 1/7,\ c_{22} = -2/21$

Thus, the value of $\bar{b}_2 = 3$ and $\beta_2 = 3/7$

Since, $\quad c_{21} = \bar{c}_{21}^{+} + \bar{c}_{21}^{-}$ and $c_{22} = \bar{c}_{22}^{+} + \bar{c}_{22}^{-}$,

$\bar{c}_{21}^{+} = 0, \bar{c}_{21}^{-} = -1/7$ since $\bar{c}_{21}$ is negative

$\bar{c}_{22}^{+} = 2/21, \bar{c}_{22}^{-} = 0$ since $\bar{c}_{22}$ is positive

Example Problem (MILP) …contd.

Thus, Gomory constraint can be written as

$$s_i - \sum_{j=1}^{m} \bar{c}_{ij}^{+} y_j - \frac{\beta_i}{\beta_i - 1} \sum_{j=1}^{m} \bar{c}_{ij}^{-} y_j = -\beta_i$$

$$i.e.,\ s_2 - \tfrac{2}{21} y_2 - \tfrac{3}{28} y_1 = -\tfrac{3}{7}$$

Insert this constraint as a new row to the previous table and solve it using Dual Simplex method

Example Problem (MILP) …contd.

Iteration	Basis	Z	x_1	x_2	y_1	y_2	s_2	b_r
4	Z	1	0	0	$\frac{7}{8}$	0	1	$\frac{33}{2}$
	x_1	0	1	0	$\frac{3}{8}$	0	1	$\frac{9}{2}$
	x_2	0	0	1	$-\frac{1}{4}$	0	1	3
	y_2	0	0	0	$\frac{9}{8}$	1	$-\frac{21}{2}$	$\frac{9}{2}$

Optimum value of Z is $\frac{33}{2}$ and the values of basic variables are $x_1 = 4.5;\ x_2 = 3;\ y_2 = 4.5$ and that of non-basic variables are zero.

This solution is satisfying all the constraints an hence the desired.

Thank You