Integer Programming

All Integer Linear Programming

D Nagesh Kumar, IISc

Objectives

- To discuss the need for Integer Programming (IP)
- To discuss about the types of IP
- To explain Integer Linear Programming (ILP)
- To discuss the Gomory Cutting Plane method for solving ILP
  - Graphically
  - Theoretically

Introduction

- In many practical problems, the values of decision variables are constrained to take only integer values
  - For example, in minimization of labor needed in a project, the number of labourers should be an integer value
- By rounding off a real value to an integer value have several fundamental problems like
  - Rounded solutions may not be feasible
  - Even if the solutions are feasible, the objective function given by the rounded off solutions may not be the optimal one
  - Finally, even if the above two conditions are satisfied, checking all the rounded-off solutions is computationally expensive (2^n possible solutions to be considered for an *n* variable problem)
- This demands the need for ***Integer Programming***

Types of IP

- ***All Integer Programming:***
  - All the variables are restricted to take only integer values
- ***Discrete Programming:***
  - All the variables are restricted to take only discrete values
- ***Mixed Integer or Discrete Programming:***
  - Only some variables are restricted to take integer or discrete values
- ***Zero – One Programming:***
  - Variables are constrained to take values of either zero or 1

Integer Linear Programming (ILP)

- An extension of linear programming (LP)
- Additional constraint: Variables should be integer valued
- Standard form of an ILP:

$$
\begin{aligned}
\max \quad & c^T X \\
\textit{subject to} \quad & AX \le b \\
& X \ge 0
\end{aligned}
$$

X must be integer valued

- Associated linear program, dropping the integer restrictions, is called *linear relaxation (LR)*

Checks for ILP:

- *Minimization*: Optimal objective value for LR is less than or equal to the optimal objective for ILP
- *Maximization*: Optimal objective value for LR is greater than or equal to that of ILP
- If LR is infeasible, then ILP is also infeasible
- If LR is optimized by integer variables, then that solution is feasible and optimal for IP

All – Integer Programming

- Most popular method: Gomory's Cutting Plane method
- Original feasible region is reduced to a new feasible region by including additional constraints such that all vertices of the new feasible region are now integer points
- Thus, an extreme point of the new feasible region becomes an optimal solution after accounting for the integer constraints
- Consider the optimization problem

$$\text{Maximize} \quad Z = 3x_1 + x_2$$

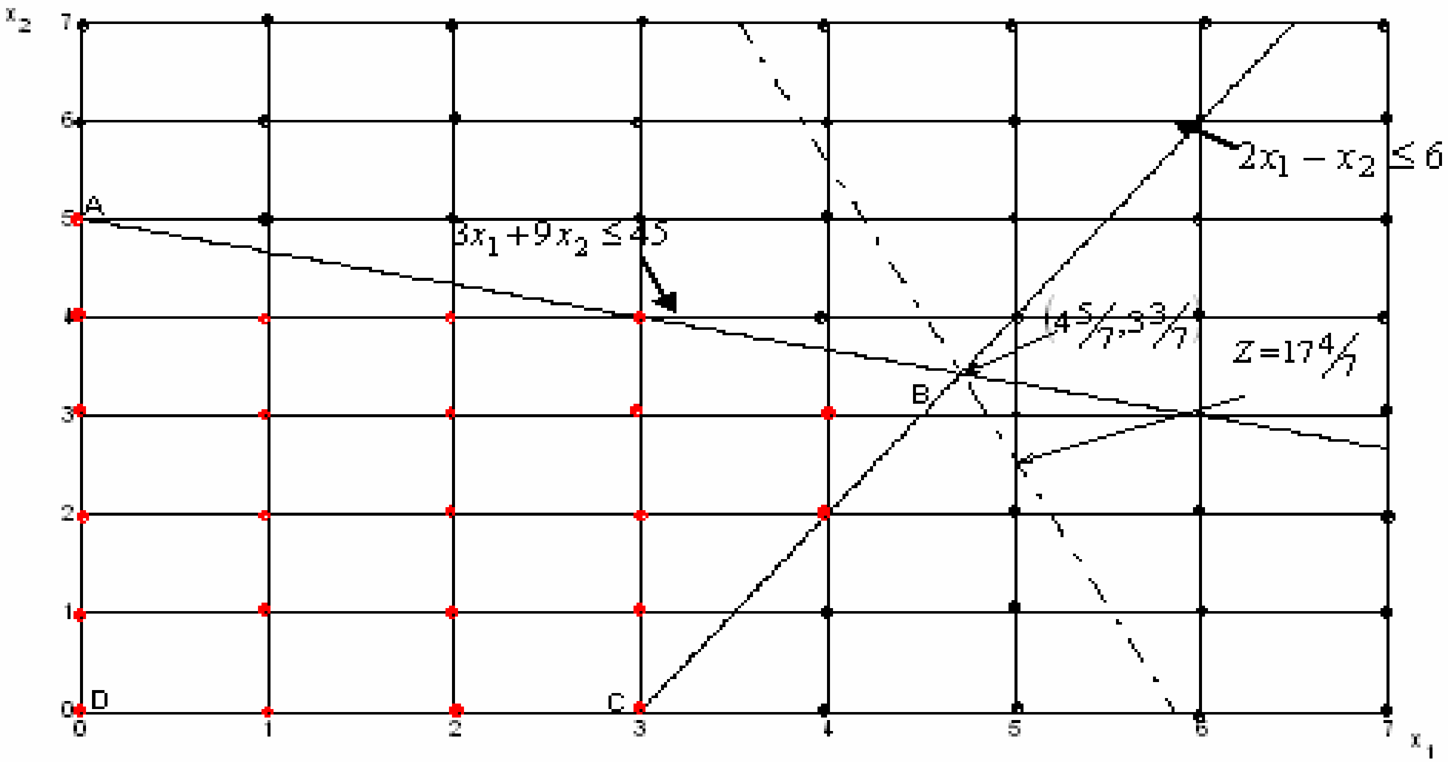
$$\text{subject to} \quad 2x_1 - x_2 \leq 6$$

$$3x_1 + 9x_2 \leq 45$$

$$x_1, x_2 \geq 0 \ ; \qquad x_1 \text{ and } x_2 \text{ are integers}$$

Graphical Illustration

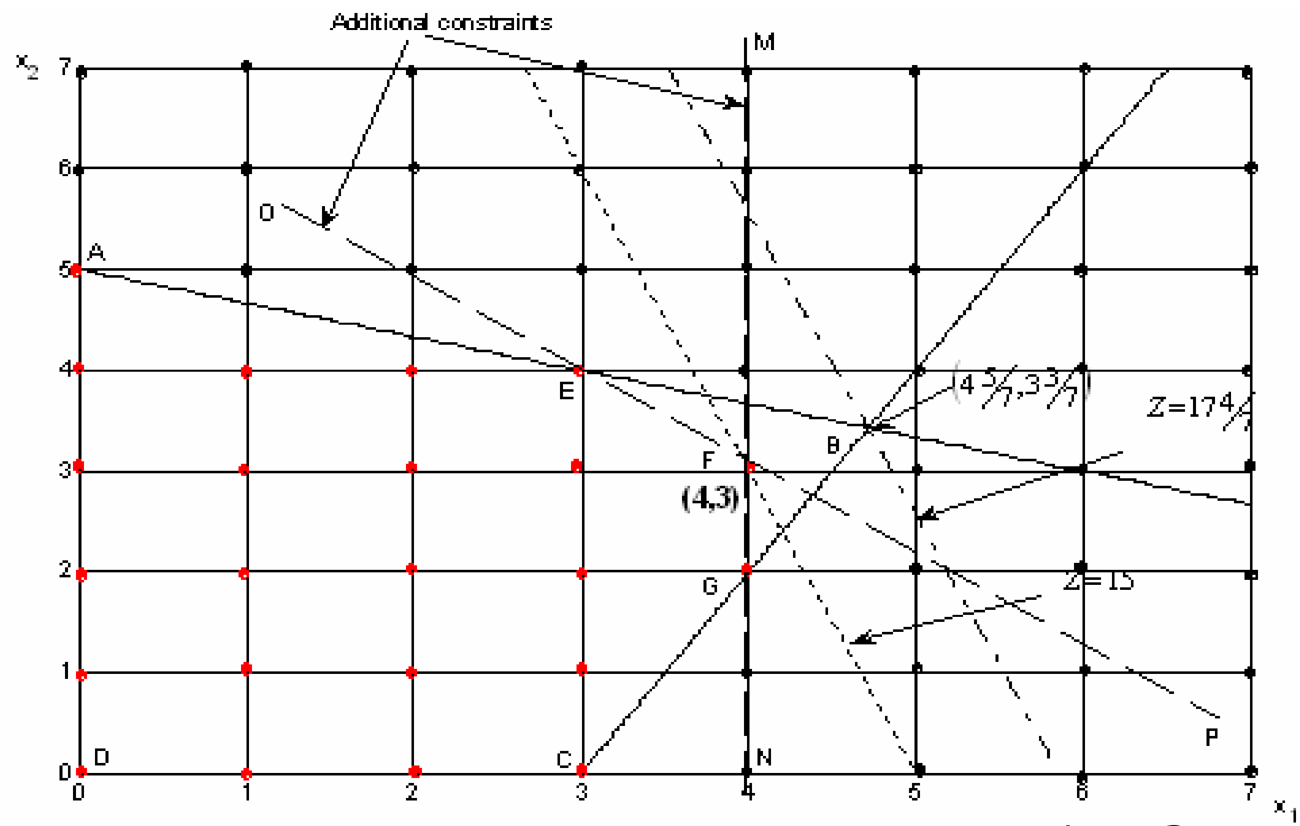
Graphical solution for the linear approximation (neglecting the integer requirements) is shown in figure



Graphical Illustration …contd.

- Optimal value of $Z = 17\frac{4}{7}$ and the solution is $x_1 = 4\frac{5}{7}, x_2 = 3\frac{3}{7}$
- Red dots in the figure show the feasible solutions accounting for the integer requirements
- These points are called integer lattice points
- Now to reduce the original feasible region to a new feasible region (considering x_1 and x_2 as integers) is done by including additional constraints
- Graphical solution for the IP is shown in figure below
- Two additional constraints (MN and OP) are included so that the original feasible region ABCD is reduced to a new feasible region AEFGCD

Graphical Illustration …contd.



Optimal value of ILP is $Z = 15$ and the solution is $x_1 = 4, x_2 = 3$

Generation of Gomory Constraints

- Consider the final tableau of an LP problem consisting of *n* basic variables (original variables) and *m* non basic variables (slack variables)
- The basic variables are represented as x_i (i=1,2,…,n) and the non basic variables are represented as y_j (j=1,2,…,m).

Basis	Z	Variables													b_r
		x_1	x_2	…	x_i	…	x_n	y_1	y_2	…	y_j	…	y_m		
Z	1	0	0		0		0	c_1	c_2		c_j		c_m		b
x_1	0	1	0		0		0	c_{11}	c_{12}		c_{1j}		c_{1m}		b_1
x_2	0	0	1		0		0	c_{21}	c_{22}		c_{2j}		c_{2m}		b_2
⋮															
x_i	0	0	0		1		0	c_{31}	c_{32}		c_{3j}		c_{3m}		b_i
⋮															
x_n	0	0	0		0		1	c_{41}	c_{42}		c_{4j}		c_{4m}		b_n

Generation of Gomory Constraints …contd.

- Pick the variable x_i having the highest fractional value. In case of a tie, choose arbitrarily any variable as x_i
- From the i[th] equation,

$$x_i = b_i - \sum_{j=1}^{m} c_{ij} y_j$$

- Expressing both b_j and c_{ij} as an integer part plus a fractional part,

$$b_i = \overline{b_i} + \beta_i$$

$$c_{ij} = \overline{c}_{ij} + \alpha_{ij}$$

Generation of Gomory Constraints …contd.

- $\bar{b}_i, \bar{c}_{ij}$ denote the integer part and
- β_i , α_{ij} denote the fractional part for which $(0 < \beta_i < 1)$ and $(0 \le \alpha_{ij} < 1)$
- Thus, the equation becomes,

$$\beta_i - \sum_{j=1}^{m} \alpha_{ij} y_j = x_i - \bar{b}_i - \sum_{j=1}^{m} \bar{c}_{ij} y_j$$

Generation of Gomory Constraints …contd.

- Considering the integer reuirements, the RHS of the equation also should be an integer.
- Thus, we have $$\left(\beta_i - \sum_{j=1}^{m} \alpha_{ij} y_j\right) \le \beta_i < 1$$
- Hence, the constraint can be expressed as,
$$\beta_i - \sum_{j=1}^{m} \alpha_{ij} y_j \le 0$$
- After introducing a slack variable s_i, the final Gomory constraint can be written as, $$s_i - \sum_{j=1}^{m} \alpha_{ij} y_j = -\beta_i$$

Procedure for solving All-Integer LP

- Solve the problem as an ordinary LP problem neglecting the integer requirements.
- If the optimum values of the variables are not integers, then choose the basic variable which has the largest fractional value, and generate Gomory constraint for that variable.
- Insert a new row with the coefficients of this constraint, to the final tableau of the ordinary LP problem.
- Solve this by applying the dual simplex method

Procedure for solving All-Integer LP ...contd.

- Check whether the new solution is all-integer or not.
- If all values are not integers, then a new Gomory constraint is developed for the non-integer valued variable from the new simplex tableau and the dual simplex method is applied again.
- The process is continued until
  - An optimal integer solution is obtained or
  - It shows that the problem has no feasible integer solution.

Thank You