Dynamic Programming Applications

Water Allocation

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Introduction and Objectives

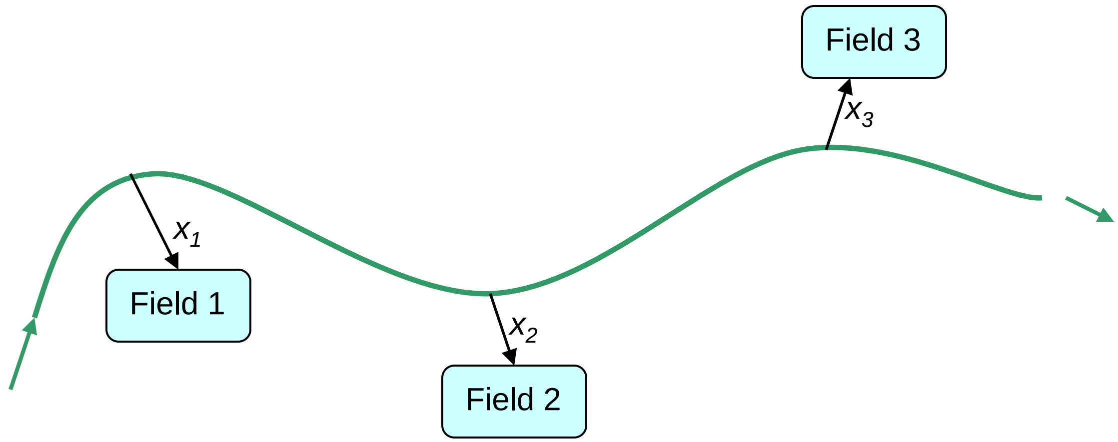
- *Dynamic Programming :* Sequential or multistage decision making process
- *Water Allocation problem* is solved as a sequential process using dynamic programming

Objectives

- To discuss the Water Allocation Problem
- To explain and develop recursive equations for backward approach
- To explain and develop recursive equations for forward approach

Water Allocation Problem

- Consider a canal supplying water for three different crops
- Maximum capacity of the canal is Q units of water.
- Amount of water allocated to each field as x_i



Water Allocation Problem … contd.

- Net benefits from producing the crops can be expressed as a function of the water allotted.

$$NB_1(x_1) = 5x_1 - 0.5x_1^{\,2}$$

$$NB_2(x_2) = 8x_2 - 1.5x_2^{\,2}$$

$$NB_3(x_3) = 7x_3 - x_3^{\,2}$$

- Optimization Problem: Determine the optimal allocations x_i to each crop that maximizes the total net benefits from all the three crops

Solution using Dynamic Programming

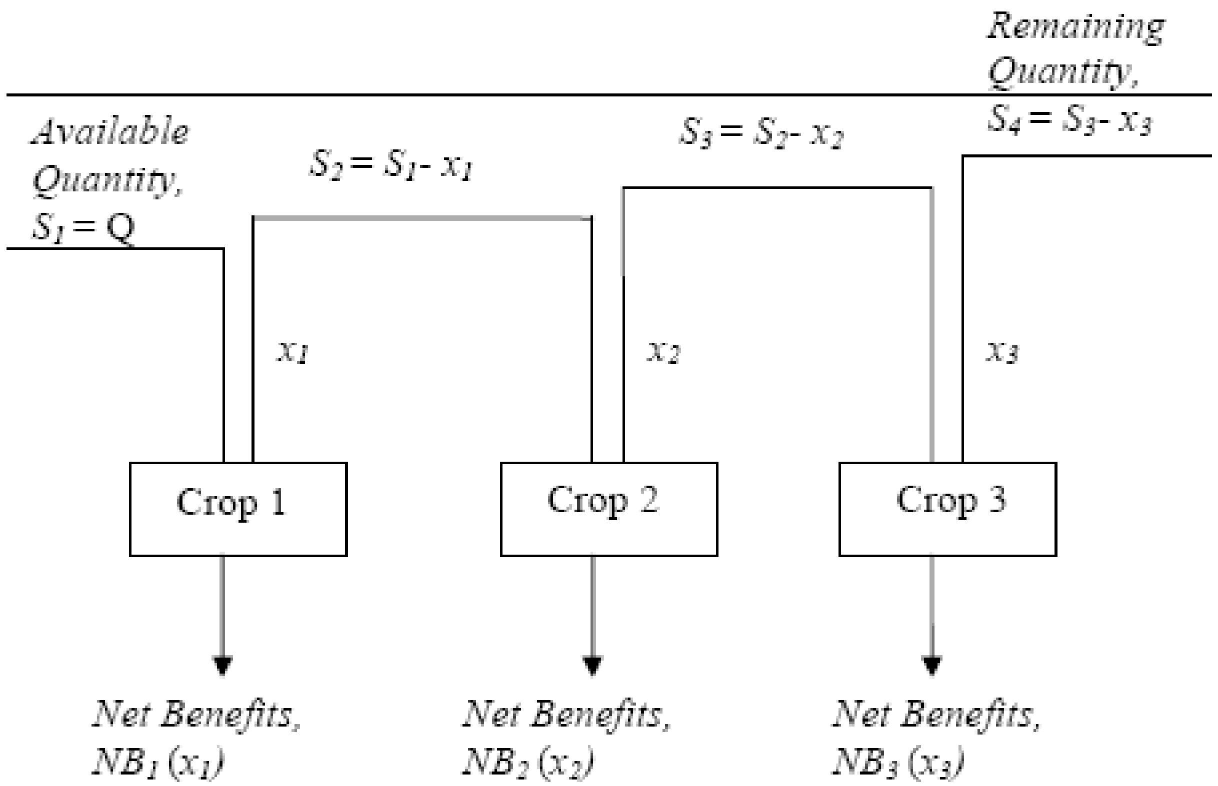
- Structure the problem as a sequential allocation process or a multistage decision making procedure.
- Allocation to each crop is considered as a decision stage in a sequence of decisions.
- Amount of water allocated to crop *i* is x_i
- Net benefit from this allocation is $NB_i(x_i)$
- Introduce one state variable S_i :- Amount of water available to the remaining *(3-i)* crops
- State transformation equation can be written as

$$S_{i+1} = S_i - x_i$$

defines the state in the next stage

Sequential Allocation Process

The allocation problem is shown as a sequential process



Backward Recursive Equations

- Objective function: To maximize the net benefits

$$\max \sum_{i=1}^{3} NB_i(x_i)$$

- Subjected to the constraints

$$x_1 + x_2 + x_3 \leq Q$$
$$0 \leq x_i \leq Q \qquad for\ i = 1,2,3$$

- Let $f_1(Q)$ be the maximum net benefits that can be obtained from allocating water to crops 1,2 and 3

$$f_1(Q) = \max_{\substack{x_1 + x_2 + x_3 \leq Q \\ x_1, x_2, x_3 \geq 0}} \left[\sum_{i=1}^{3} NB_i(x_i) \right]$$

Backward Recursive Equations … contd.

- Transforming this into three problems each having only one decision variable

$$f_1(Q) = \max_{\substack{x_1 \\ 0 \le x_1 \le Q}} \left[NB_1(x_1) + \max_{\substack{x_2 \\ 0 \le x_2 \le Q - x_1 = S_2}} \left\{ NB_2(x_2) + \max_{\substack{x_3 \\ 0 \le x_3 \le S_2 - x_2 = S_3}} NB_3(x_3) \right\} \right]$$

- Now starting from the last stage, let $f_3(S_3)$ be the maximum net benefits from crop 3.
- State variable S_3 for this stage can vary from 0 to Q
- Thus,

$$f_3(S_3) = \max_{\substack{x_3 \\ 0 \le x_3 \le S_3}} NB_3(x_3)$$

Backward Recursive Equations … contd.

- But $S_3 = S_2 - x_2$. Therefore $f_3(S_3) = f_3(S_2 - x_2)$
- Hence,

$$f_1(Q) = \max_{\substack{x_1 \\ 0 \le x_1 \le Q}} \left[NB_1(x_1) + \max_{\substack{x_2 \\ 0 \le x_2 \le Q - x_1 = S_2}} \{ NB_2(x_2) + f_3(S_2 - x_2) \} \right]$$

- Now, let $f_2(S_2)$ be the maximum benefits derived from crops 2 and 3 for a given quantity S_2 which can vary between 0 and Q
- Therefore,

$$f_2(S_2) = \max_{\substack{x_2 \\ 0 \le x_2 \le Q - x_1 = S_2}} \{ NB_2(x_2) + f_3(S_2 - x_2) \}$$

Backward Recursive Equations … contd.

- Now since $S_2 = Q - x_1$, $f_1(Q)$ can be rewritten as

$$f_1(Q) = \max_{\substack{x_1 \\ 0 \le x_1 \le Q}} \left[NB_1(x_1) + f_2(Q - x_1) \right]$$

- Once the value of $f_3(S_3)$ is calculated, the value of $f_2(S_2)$ can be determined, from which $f_1(Q)$ can be determined.

Forward Recursive Equations

- Let the function $f_i(S_i)$ be the total net benefit from crops *1* to *i* for a given input of S_i which is allocated to those crops.
- Considering the first stage,

$$f_1(S_1) = \max_{\substack{x_1 \\ x_1 \le S_1}} NB_1(x_1)$$

- Solve this equation for a range of S_1 values from *0* to *Q*
- Considering the first two crops, for an available quantity of S_2, $f_2(S_2)$ can be written as

$$f_2(S_2) = \max_{\substack{x_2 \\ x_2 \le S_2}} \left[NB_2(x_2) + f_1(S_2 - x_2) \right]$$

Forward Recursive Equations … contd.

- S_2 ranges from *0* to *Q*
- Considering the whole system, $f_3(S_3)$ can be expressed as,

$$f_3(S_3) = \max_{\substack{x_3 \\ x_3 \le S_3 = Q}} \left[NB_3(x_3) + f_2(S_3 - x_3) \right]$$

- If the whole *Q* units of water should be allocated then the value of S_3 can be taken as equal to *Q*
- Otherwise, S_3 will take a range of values from *0* to *Q*

Conclusion

The basic equations for the water allocation problem using both the approaches are discussed

A numerical problem and its solution will be described in the next lecture

Thank You