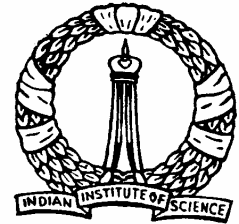


Dynamic Programming

Introduction



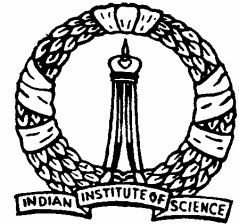
Introduction and Objectives

Introduction

- *Complex problems are sometimes solved quickly if approached in a sequential manner*
- *Dynamic Programming : Sequential or multistage decision making process*
- *Basic approach behind dynamic programming: Solution is found out in multi stages*
- *Works in a “divide and conquer” manner*

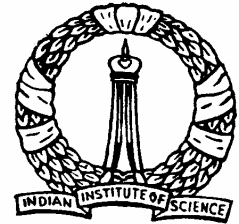
Objectives

- To discuss the Sequential Optimization and Multistage Decision Process
- To represent a Multistage Decision Process
- To describe the concept of sub-optimization and principle of optimality



Sequential Optimization

- Problem is divided into smaller sub-problems
- Optimize these sub-problems without losing the integrity of the original problem
- Decisions are made sequentially
- Also called multistage decision problems since decisions are made at a number of stages
- A N variable problem is represented by N single variable problems
- These problems are solved successively to get the optimal value of the original problem



Sequential Optimization ...contd.

- Serial multistage decision process: Single variable problems are connected in series so that the output of one stage will be the input to the succeeding stage
- For example, consider a water allocation problem to N users
- The objective function is to maximize the total net benefit from all users
- The problem can be solved by considering each user separately and optimizing the individual net benefits, subject to constraints and then adding up the benefits from all users to get the total optimal benefit

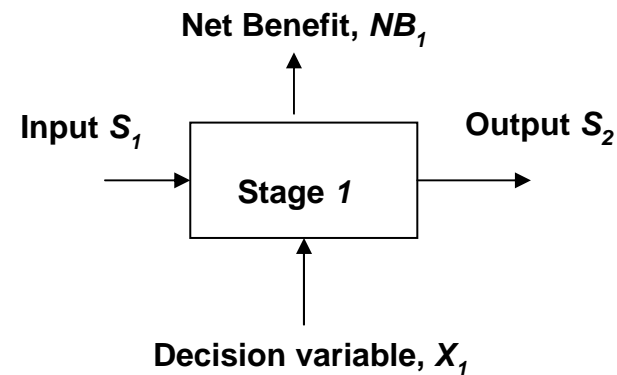


Representation of Multistage Decision Process

➤ Consider a single stage decision process

➤ Here

- S_1 is the input state variable
- S_2 is the output state variable
- X_1 is the decision variables and
- NB_1 is the net benefit



➤ The transformation function for the input and output is

$$S_2 = g(X_1, S_1)$$

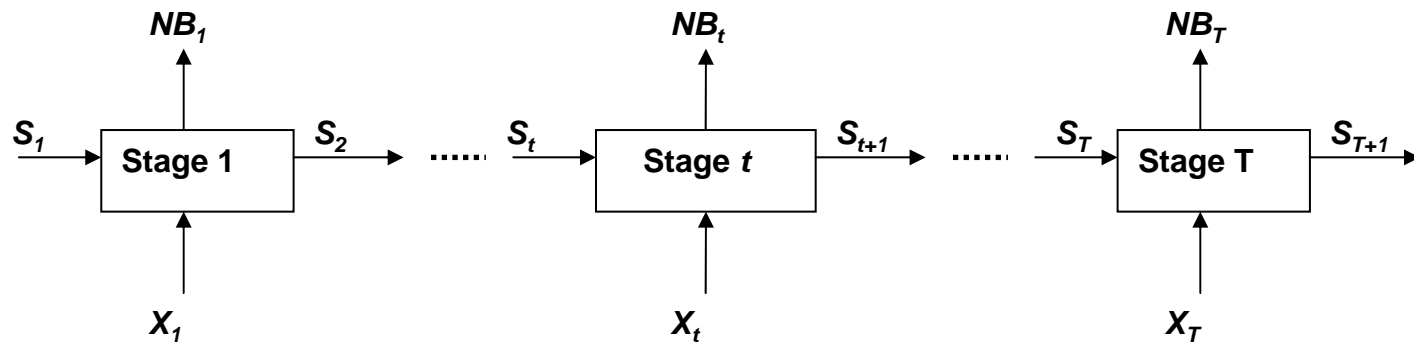
➤ Net benefits are expressed as a function of decision variables and input variable

$$NB_1 = h(X_1, S_1)$$



Representation of Multistage Decision Process ...contd.

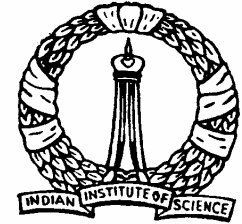
- Now, consider a serial multistage decision process consisting of T stages



- For the t^{th} stage the stage transformation and the benefit functions are

$$S_{t+1} = g(X_t, S_t)$$

$$NB_t = h(X_t, S_t)$$



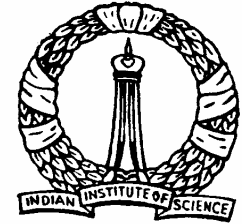
Representation of Multistage Decision Process ...contd.

- Objective of this multistage problem is to find the optimum values of all decision variables X_1, X_2, \dots, X_T such that the individual net benefits of each stage that is expressed by some objective function, $f(NB_t)$ and the total net benefit which is expressed by $f(NB_1, NB_2, \dots, NB_T)$ should be maximized
- Dynamic programming can be applied to this multistage problem if the objective function is separable and monotonic
- An objective function is separable, if it can be decomposed and expressed as a sum or product of individual net benefits of each stage

$$f = \sum_{t=1}^T NB_t = \sum_{t=1}^T h(X_t, S_t)$$

or

$$f = \prod_{t=1}^T NB_t = \prod_{t=1}^T h(X_t, S_t)$$



Representation of Multistage Decision Process ...contd.

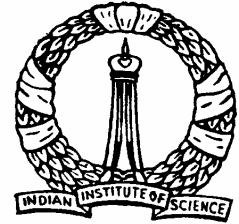
- An objective function is monotonic if for all values of a and b for which the value of the benefit function is ,

$$h(x_t = a, S_t) \geq h(x_t = b, S_t)$$

then,

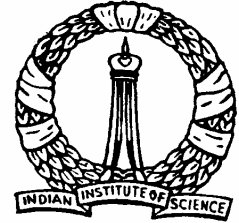
$$f(x_1, x_2, \dots, x_t = a, \dots, x_T, S_{t+1}) \geq f(x_1, x_2, \dots, x_t = b, \dots, x_T, S_{t+1})$$

is satisfied



Types of Multistage Decision Process

- A serial multistage problem can be divided into three categories
 - *Initial value problem*
 - Value of the initial state variable, S_1 is given
 - *Final value problem*
 - Value of the final state variable, S_T is given
 - Final value problem can be transformed into an initial value problem by reversing the direction of the state variable, S_t
 - *Boundary value problem.*
 - Values of both the initial and final state variables, S_1 and S_T are given



Concept of Sub-Optimization

Consider the objective function consisting of T decision variables x_1, x_2, \dots, x_T

$$f = \sum_{t=1}^T NB_t = \sum_{t=1}^T h(X_t, S_t)$$

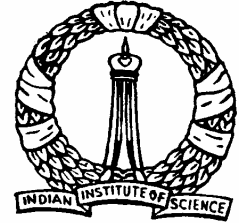
satisfying the equations,

$$S_{t+1} = g(X_t, S_t)$$

$$NB_t = h(X_t, S_t)$$

$$\text{for } t = 1, 2, \dots, T$$

To solve this problem through dynamic programming, the concepts of sub-optimization and principle of optimality are used



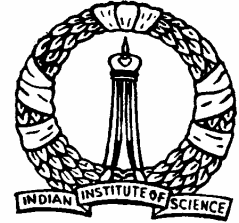
Concept of Sub-Optimization ...contd.

- Consider the design of a water tank in which the cost of construction is to be minimized
- Capacity of the tank to be designed is K
- Main components of a water tank include (i) tank (ii) columns to support the tank and (iii) the foundation
- Optimization can be done by breaking the system into individual parts and optimizing each part separately
- While breaking and sub-optimizing, a logical procedure should be used; otherwise this approach can lead to a poor solution



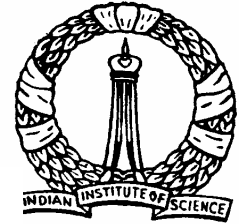
Concept of Sub-Optimization ...contd.

- For example, sub-optimization of columns without considering the other two components may cause the use of heavy concrete columns with less reinforcement, since the cost of steel is high
- But while considering the sub-optimization of foundation component, the cost becomes higher as the foundation should be strong enough to carry these heavy columns
- Thus, the sub-optimization of columns before considering the sub-optimization of foundation will adversely affect the overall design

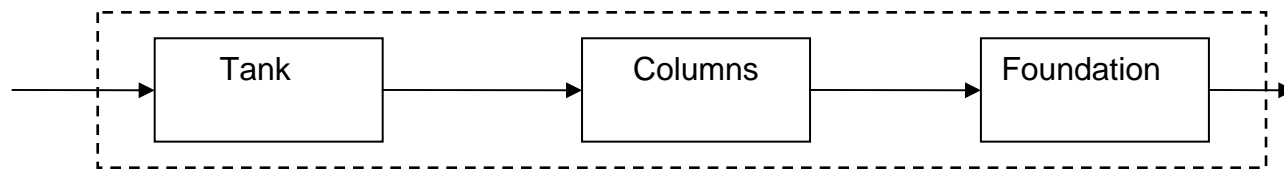
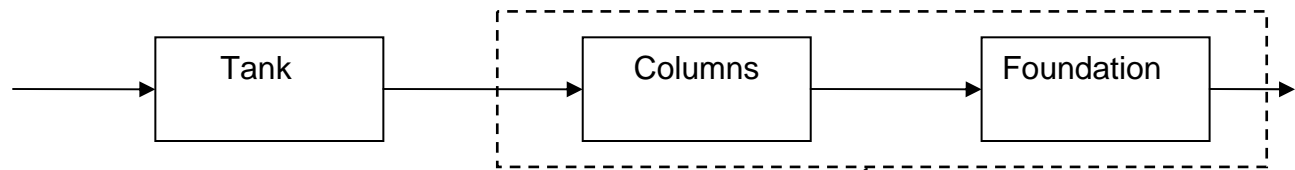
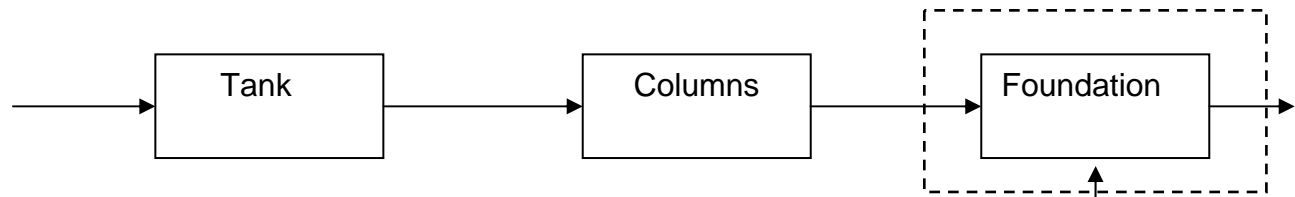
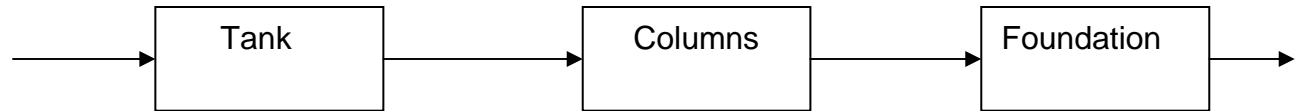


Concept of Sub-Optimization ...contd.

- In most of the serial systems sub-optimization can start from the last component (or first component) since it does not influence the other components
- Thus, for the above problem, foundation can be suboptimized independently
- Then, the last two components (columns and foundation) are considered as a single component and sub-optimization is done without affecting other components
- This process can be repeated for any number of end components



Process of sub-optimization





Principle of Optimality

- Belman stated the principle of optimality which explains the process of multi stage optimization as:
 - *“An optimal policy (or a set of decisions) has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.”*



Thank You