Linear Programming Applications

Assignment Problem

D Nagesh Kumar, IISc

Introduction

- Assignment problem is a particular class of transportation linear programming problems
- Supplies and demands will be integers (often 1)
- Traveling salesman problem is a special type of assignment problem

Objectives

- To structure and formulate a basic assignment problem
- To demonstrate the formulation and solution with a numerical example
- To formulate and solve traveling salesman problem as an assignment problem

Structure of Assignment Problem

- Assignment problem is a special type of transportation problem in which
  - Number of supply and demand nodes are equal.
  - Supply from every supply node is one.
  - Every demand node has a demand for one.
  - Solution is required to be all integers.

Structure of Assignment Problem …contd.

- ***Goal of an general assignment problem:*** Find an optimal assignment of machines (laborers) to jobs without assigning an agent more than once and ensuring that all jobs are completed
- The objective might be to minimize the total time to complete a set of jobs, or to maximize skill ratings, maximize the total satisfaction of the group or to minimize the cost of the assignments
- This is subjected to the following requirements:
  - Each machine is assigned no more than one job.
  - Each job is assigned to exactly one machine.

Formulation of Assignment Problem

- Consider m laborers to whom n tasks are assigned
- No laborer can either sit idle or do more than one task
- Every pair of person and assigned work has a rating
- Rating may be cost, satisfaction, penalty involved or time taken to finish the job
- N^2 such combinations of persons and jobs assigned
- Optimization problem: Find such job-man combinations that optimize the sum of ratings among all.

Formulation of Assignment Problem …contd.

- Representation of this problem as a special case of transportation problem
  - *laborers* as *sources*
  - *tasks* as *destinations*
  - Supply available at each source is 1
  - Demand required at each destination is 1
  - Cost of assigning (transporting) laborer i to task j is $c_{ij.}$
- It is necessary to first balance this problem by adding a dummy laborer or task depending on whether $m < n$ or $m > n$, respectively
- Cost coefficient c_{ij} for this dummy will be zero.

Formulation of Assignment Problem …contd.

- Let x_{ij} be the decision variable

$$x_{ij} = \begin{cases} 0, \; if \;\; the \;\; j^{th} \;\; job \; is \; not \; assigned \; to \; the \; i^{th} \; machine \\ 1, \; if \;\; the \;\; j^{th} \;\; job \; is \; assigned \; to \; the \; i^{th} \; machine \end{cases}$$

- The objective function is

$$Minimize \;\; \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Formulation of Assignment Problem …contd.

- Since each task is assigned to exactly one laborer and each laborer is assigned only one job, the constraints are

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad for \;\; j = 1, 2, \ldots n$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad for \;\; i = 1, 2, \ldots m$$

$$x_{ij} = 0 \;\; or \;\; 1$$

- Due to the special structure of the assignment problem, the solution can be found out using a more convenient method called Hungarian method.

Example (1)

- Consider three jobs to be assigned to three machines. The cost for each combination is shown in the table below. Determine the minimal job – machine combinations

Table 1

Job	Machine			
	1	2	3	a_i
1	5	7	9	1
2	14	10	12	1
3	15	13	16	1
b_j	1	1	1	

Example (1)… contd.

<u>Solution:</u>

Step 1:

- Create zero elements in the cost matrix (zero assignment) by subtracting the smallest element in each row (column) from the corresponding row (column).
- Considering the rows first, the resulting cost matrix is obtained by subtracting 5 from row 1, 10 from row 2 and 13 from row 3

<u>Table 2</u>

	1	2	3
1	0	2	4
2	4	0	2
3	2	0	3

Example (1)… contd.

Step 2:

- Repeating the same with columns, the final cost matrix is

Table 3

	1	2	3
1	*0*	2	2
2	4	0	*0*
3	2	*0*	3

- The italicized zero elements represent a feasible solution
- Thus the optimal assignment is (1,1), (2,3) and (3,2)
- The total cost is equal to (5 +12+13) = 60

Example (2)

- In the above example, it was possible to obtain the feasible assignment
- But in more complicate problems, additional rules are required which are explained in the next example.

Example 2 (Taha, 1982)

- Consider four jobs to be assigned to four machines. Determine the minimal job – machine combinations.

Example (2) …contd.

- The cost for each combination is shown in the table below

Table 4

Job	Machine				
	1	2	3	4	a_i
1	1	4	6	3	1
2	8	7	10	9	1
3	4	5	11	7	1
4	6	7	8	5	1
b_j	1	1	1	1	

Example (2) …contd.

Solution:

Step 1: Create zero elements in the cost matrix by subtracting the smallest element in each row from the corresponding row.

Table 5

	1	2	3	4
1	0	3	5	2
2	1	0	3	2
3	0	1	7	3
4	1	2	3	0

Example (2) …contd.

- Step 2: Repeating the same with columns, the final cost matrix is

Table 6

	1	2	3	4
1	0	3	2	2
2	1	0	0	2
3	0	1	4	3
4	1	2	0	0

- Rows 1 and 3 have only one zero element
- Both of these are in column 1, which means that both jobs 1 and 3 should be assigned to machine 1

Example (2) …contd.

- As one machine can be assigned with only one job, a feasible assignment to the zero elements is not as in the previous example
- Step 3: Draw a minimum number of lines through some of the rows and columns so that all the zeros are crossed out

Table 7

	1	2	3	4
1	0	3	2	2
2	1	0	0	2
3	0	1	4	3
4	1	2	0	0

Example (2) …contd.

- Step 4: Select the smallest uncrossed element (which is 1 here). Subtract it from every uncrossed element and also add it to every element at the intersection of the two lines.
- This will give the following table

Table 8

	1	2	3	4
1	*0*	2	1	1
2	2	0	*0*	2
3	0	*0*	3	2
4	2	2	0	*0*

Example (2) …contd.

- This gives a feasible assignment (1,1), (2,3), (3,2) and (4,4)
- And the total cost is 1+10+5+5 = 21.
- If the optimal solution had not been obtained in the last step, then the procedure of drawing lines has to be repeated until a feasible solution is achieved.

Formulation of Traveling Salesman Problem (TSP) as an Assignment Problem

- A traveling salesman has to visit n cities and return to the starting point
- He has to start from any one city and visit each city only once.
- Suppose he starts from the k^{th} city and the last city he visited is m
- Let the cost of travel from i^{th} city to j^{th} city be c_{ij}.
- Then the objective function is

$$Minimize \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Formulation of Traveling Salesman Problem (TSP) as an Assignment Problem …contd.

- subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad for \;\; j = 1, 2, \ldots n, i \neq j, i \neq m$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad for \;\; i = 1, 2, \ldots m, i \neq j, i \neq m$$

$$x_{mk} = 1$$

$$x_{ij} = 0 \;\; or \;\; 1$$

<u>Solution Procedure:</u>

- Solve the problem as an assignment problem using the method used to solve the above examples
- If the solutions thus found out are cyclic in nature, then that is the final solution

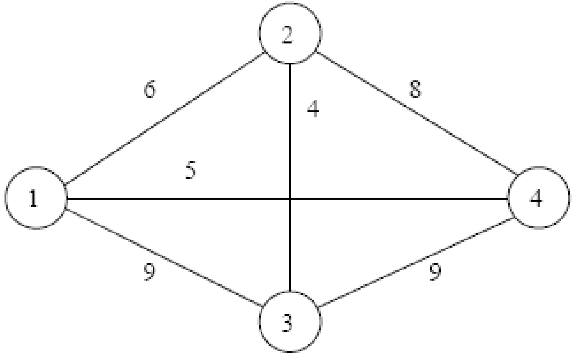
Formulation of Traveling Salesman Problem (TSP) as an Assignment Problem …contd.

Solution Procedure …contd.

- If it is not cyclic, then select the lowest entry in the table (other than zero)
- Delete the row and column of this lowest entry and again do the zero assignment in the remaining matrix
- Check whether cyclic assignment is available
- If not, include the next higher entry in the table and the procedure is repeated until a cyclic assignment is obtained.

Traveling Salesman Problem (TSP) - Example

- Consider a four city TSP for which the cost between the city pairs are as shown in the figure below. Find the tour of the salesman so that the cost of travel is minimal.



Cost matrix

Table 9

	1	2	3	4
1	∞	4	9	5
2	6	∞	4	8
3	9	4	∞	9
4	5	8	9	∞

Traveling Salesman Problem (TSP) – Example …contd.

<u>Solution:</u>

- Step 1: The optimal solution after using the Hungarian method is shown below.

<u>Table 10</u>

	1	2	3	4
1	∞	0	5	<u>*0*</u>
2	2	∞	<u>*0*</u>	3
3	5	<u>*0*</u>	∞	4
4	<u>*0*</u>	3	4	∞

- The optimal assignment is 1→ 4, 2→ 3, 3→ 2, 4→ 1 which is not cyclic

Traveling Salesman Problem (TSP) – Example …contd.

Step 2:

- Consider the lowest entry '2' of the cell (2,1)
- If there is a tie in selecting the lowest entry, then break the tie arbitrarily
- Delete the 2^{nd} row and 1^{st} column
- Do the zero assignment in the remaining matrix

Traveling Salesman Problem (TSP) – Example …contd.

- The resulting table is

Table 11

	1	2	3	4
1	∞	0	4	*0*
2	2	∞	0	3
3	5	*0*	∞	4
4	0	0	*0*	∞

- Next optimal assignment is 1→ 4, 2→1, 3→ 2, 4→ 3 which is cyclic
- Required tour is 1→ 4→3→ 2→ 1
- Optimal total travel cost is 5 + 9 + 4 + 6 = 24

Thank You