Linear Programming Applications

Transportation Problem

D Nagesh Kumar, IISc

Introduction

- Transportation problem is a special problem of its own structure.
- Planning model that allocates resources, machines, materials, capital etc. in a best possible way
- In these problems main objective generally is to minimize the cost of transportation or maximize the benefit.

Objectives

- To structure a basic transportation problem
- To demonstrate the formulation and solution with a numerical example

Structure of the Problem

- A classic transportation problem is concerned with the distribution of any commodity (resource) from any group of 'sources' to any group of destinations or 'sinks'
- The amount of resources from source to sink are the decision variables
- The criterion for selecting the optimal values of the decision variables (like minimization of costs or maximization of profits) will be the objective function
- The limitation of resource availability from sources will constitute the constraint set.

Structure of the Problem …contd.

- Let there be *m* origins namely $O_1, O_2, \ldots, O_m$ and *n* destinations $D_1, D_2, \ldots, D_n$
- Amount of commodity available in i^{th} source is $\boldsymbol{a_i}$, $i=1,2,\ldots,m$
- Demand in j^{th} sink is $\boldsymbol{b_j}$, j = 1, 2, … , n.
- Cost of transportation of unit amount of material from *i* to *j* is $\boldsymbol{c_{ij}}$
- Amount of commodity supplied from *i* to *j* be denoted as $\boldsymbol{x_{ij}}$
- Thus, the cost of transporting $\boldsymbol{x_{ij}}$ units of commodity from *i* to *j* is

$$c_{ij} \times x_{ij}$$

Structure of the Problem …contd.

- The objective of minimizing the total cost of transportation

Minimize $$f = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (1)$$

- Generally, in transportation problems, the amount of commodity available in a particular source should be equal to the amount of commodity supplied from that source. Thus, the constraint can be expressed as

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, \ldots, m \qquad (2)$$

Structure of the Problem …contd.

- Total amount supplied to a particular sink should be equal to the corresponding demand

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, \ldots, n \qquad (3)$$

- The set of constraints given by eqns (2) and (3) are consistent only if total supply and total demand are equal

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \qquad i = 1, 2, \ldots, m \;, j = 1, 2, \ldots, n \qquad (4)$$

Structure of the Problem …contd.

- But in real problems constraint (4) may not be satisfied. Then, the problem is said to be unbalanced
- Unbalanced problems can be modified by adding a fictitious (dummy) source or destination which will provide surplus supply or demand respectively
- The transportation costs from this dummy source to all destinations will be zero
- Likewise, the transportation costs from all sources to a dummy destination will be zero

Structure of the Problem… contd.

- This restriction causes one of the constraints to be redundant
- Thus the above problem have m x n decision variables and $(m + n - 1)$ equality constraints.
- The non-negativity constraints can be expressed as

$$x_{ij} \geq 0 \qquad i = 1, 2, \ldots, m \;, \quad j = 1, 2, \ldots, n \qquad (5)$$

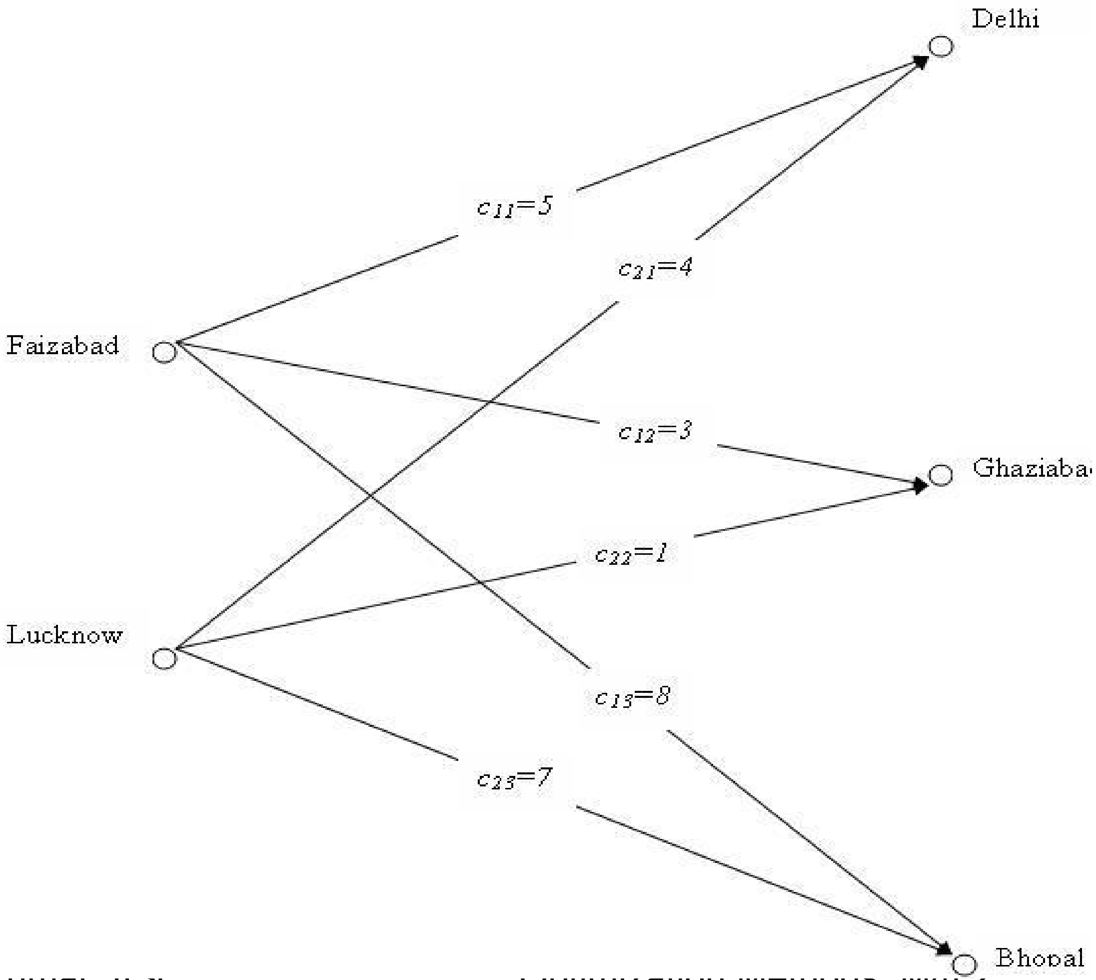
Example (1)

- Consider a transport company which has to supply 4 units of paper materials from each of the cities Faizabad and Lucknow
- The material is to be supplied to Delhi, Ghaziabad and Bhopal with demands of four, one and three units respectively
- Cost of transportation per unit of supply (c_{ij}) is indicated in the figure shown in the next slide
- Decide the pattern of transportation that minimizes the cost.

Example (1)… contd.

Solution:

- Let the amount of material supplied from source *i* to sink *j* as x_{ij}
- Here $m = 2$; $n = 3$.



Example (1)… contd.

- Total supply = 4+4 = 8 units
- Total demand = 4+1+ 3 = 8 units
- Since both are equal, the problem is balanced
- Objective function is to minimize the total cost of transportation from all combinations

  Minimize $$f = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

  i.e.

  Minimize

$$f = 5\,x_{11} + 3\,x_{12} + 8\,x_{13} + 4\,x_{21} + x_{22} + 7\,x_{23} \qquad (6)$$

Example (1)… contd.

The constraints are explained below:

- The total amount of material supplied from each source city should be equal to 4

$$\sum_{j=1}^{3} x_{ij} = 4 \; ; \quad i = 1, 2$$

i.e. $x_{11} + x_{12} + x_{13} = 4 \quad for\ i = 1$ (7)

$x_{21} + x_{22} + x_{23} = 4 \quad for\ i = 2$ (8)

Example (1)… contd.

- The total amount of material received by each destination city should be equal to the corresponding demand

$$\sum_{i=1}^{2} x_{ij} = b_j \; ; \quad j = 1, 2, 3$$

i.e.

$$x_{11} + x_{21} = 4 \qquad for\ j = 1 \qquad (9)$$

$$x_{12} + x_{22} = 1 \qquad for\ j = 2 \qquad (10)$$

$$x_{13} + x_{23} = 3 \qquad for\ j = 3 \qquad (11)$$

- Non-negativity constraints

$$x_{ij} \geq 0 \; ; \quad i = 1, 2; j = 1, 2, 3 \qquad (12)$$

Example (1)…contd.

- Optimization problem has 6 decision variables and 5 constraints
- Since the optimization model consists of equality constraints, Big M method is used to solve
- Since there are five equality constraints, introduce five artificial variables viz., R_1, R_2, R_3, R_4 and R_5.

Example (1)…contd.

The objective function and the constraints can be expressed as

Minimize $f = 5\times x_{11} + 3\times x_{12} + 8\times x_{13} + 4\times x_{21} + 1\times x_{22} + 7\times x_{23}$
$+ M\times R_1 + M\times R_2 + M\times R_3 + M\times R_4 + M\times R_5$

subject to $x_{11} + x_{12} + x_{13} + R_1 = 4$

$x_{21} + x_{22} + x_{23} + R_2 = 4$

$x_{11} + x_{21} + R_3 = 4$

$x_{12} + x_{22} + x_{23} + R_4 = 1$

$x_{13} + x_{23} + R_5 = 3$

Example (1)….contd.

➢ Modifying the objective function to make the coefficients of the artificial variable equal to zero, the final form objective function is

$$
\begin{aligned}
f &+ (-5+2M)\times x_{11} + (-3+2M)\times x_{12} + (-8+2M)\times x_{13} \\
&+ (-4+2M)\times x_{21} + (-1+2M)\times x_{22} + (-7+2M)\times x_{23} \\
&- 0\times R_1 + 0\times R_2 + 0\times R_3 + 0\times R_4 + 0\times R_5
\end{aligned}
$$

The solution of the model using simplex method is shown in the following slides

Example (1)…contd.

First iteration

Table 1

First iteration

Basic variables	Variables											RHS	Ratio
	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	R_1	R_2	R_3	R_4	R_5		
Z	-5 +2M	-3 +2M	-8 +2M	-4 +2M	-1 +2M	-7 +2M	0	0	0	0	0	16M	
R_1	1	1	1	0	0	0	1	0	0	0	0	4	-
R_2	0	0	0	1	1	1	0	1	0	0	0	4	4
R_3	1	0	0	1	0	0	0	0	1	0	0	4	-
R_4	0	1	0	0	1	0	0	0	0	1	0	1	1
R_5	0	0	1	0	0	1	0	0	0	0	1	3	-

Example (1)…contd.

Second iteration

Table 2

Second iteration

Basic variables	Variables											RHS	Ratio
	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	R_1	R_2	R_3	R_4	R_5		
Z	-5+2M	-1	-8+2M	-4+2M	0	-7+2M	0	0	0	1-2M	0	1+14M	-
R_1	1	1	1	0	0	0	1	0	0	0	0	4	-
R_2	0	-1	0	1	0	0	0	1	0	-1	0	3	3
R_3	1	0	0	1	0	0	0	0	1	0	0	4	4
X_{22}	0	1	0	0	1	1	0	0	0	1	0	1	-
R_5	0	0	1	0	0	1	0	0	0	0	1	3	-

Example (1)…contd.

Third iteration

Table 3

Third iteration

Basic variables	Variables											RHS	Ratio
	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	R_1	R_2	R_3	R_4	R_5		
Z	-5+2M	-5+2M	-8+2M	0	0	-7+2M	0	4-2M	0	-3	0	13+8M	-
R_1	1	1	1	0	0	0	1	0	0	0	0	4	4
X_{21}	0	-1	0	1	0	0	0	1	0	-1	0	3	-
R_3	1	1	0	0	0	0	0	-1	1	1	0	1	1
X_{22}	0	1	0	0	1	1	0	0	0	1	0	1	-
R_5	0	0	1	0	0	1	0	0	0	0	1	3	-

Example (1)…contd.

Fourth iteration

Table 4

Fourth iteration

Basic variables	Variables											RHS	Ratio
	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	R_1	R_2	R_3	R_4	R_5		
Z	0	0	-8+2M	0	0	-7+2M	0	-1	5-2M	2-2M	0	18+6M	-
R_1	0	0	1	0	0	0	1	1	-1	-1	0	3	-
X_{21}	0	-1	0	1	0	0	0	1	0	-1	0	3	-
X_{11}	1	1	0	0	0	0	0	-1	1	1	0	1	-
X_{22}	0	1	0	0	1	1	0	0	0	1	0	1	1
R_5	0	0	1	0	0	1	0	0	0	0	1	3	3

Example (1)…contd.

- Repeating the same procedure

  Minimum total cost,

$$f = 42$$

  Optimum decision variable values:

$$x_{11} = 2.2430,\ \ x_{12} = 0.00,\ \ x_{13} = 1.7570,$$

$$x_{21} = 1.7570,\ \ x_{22} = 1.00,\ \ x_{23} = 1.2430.$$

Example (2)

- Consider three factories (F) located in three different cities, producing a particular chemical. The chemical is to get transported to four different warehouses (Wh), from where it is supplied to the customers. The transportation cost per truck load from each factory to each warehouse is determined and are given in the table below. Production and demands are also given in the table below.

	Wh1	Wh2	Wh3	Wh4	Production
F1	523	682	458	850	60
F2	420	412	362	729	110
F3	670	558	895	695	150
Demand	65	85	80	70	

Example (2) …contd.

<u>Solution:</u>

- Let the amount of chemical to be transported from factory *i* to warehouse *j* be x_{ij}.
- Total supply = 60+110+150 = 320
- Total demand = 65+85+80+70 = 300
- Since the total demand is less than total supply, add one fictitious ware house, Wh5 with a demand of 20.
- Thus, $m = 3; n = 5$

Example (2) …contd.

- The modified table is shown below

	Wh1	Wh2	Wh3	Wh4	Wh5	Production
F1	523	682	458	850	0	60
F2	420	412	362	729	0	110
F3	670	558	895	695	0	150
Demand	65	85	80	70	20	

Example (2) …contd.

- Objective function is to minimize the total cost of transportation from all combinations

Minimize $f = 523\, x_{11} + 682\, x_{12} + 458\, x_{13} + 850\, x_{14} + 0\, x_{15} + 420\, x_{21}$

$+ 412\, x_{22} + 362\, x_{23} + 729\, x_{24} + 0\, x_{25} + 670\, x_{31}$

$+ 558\, x_{32} + 895\, x_{33} + 695\, x_{34} + 0\, x_{35}$

subject to the constraints

$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 60$ *for* $i = 1$

$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 110$ *for* $i = 2$

$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 150$ *for* $i = 3$

Example (2) …contd.

$$x_{11} + x_{21} + x_{31} = 65 \qquad \text{for } j = 1$$

$$x_{12} + x_{22} + x_{32} = 85 \qquad \text{for } j = 2$$

$$x_{13} + x_{23} + x_{23} = 90 \qquad \text{for } j = 3$$

$$x_{14} + x_{24} + x_{24} = 80 \qquad \text{for } j = 4$$

$$x_{15} + x_{25} + x_{25} = 20 \qquad \text{for } j = 5$$

$$x_{ij} \geq 0 \qquad i = 1, 2,3; j=1, 2, 3,4$$

This optimization problem can be solved using the same procedure used for the previous problem.

Thank You