Linear Programming

Other Algorithms

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Introduction & Objectives

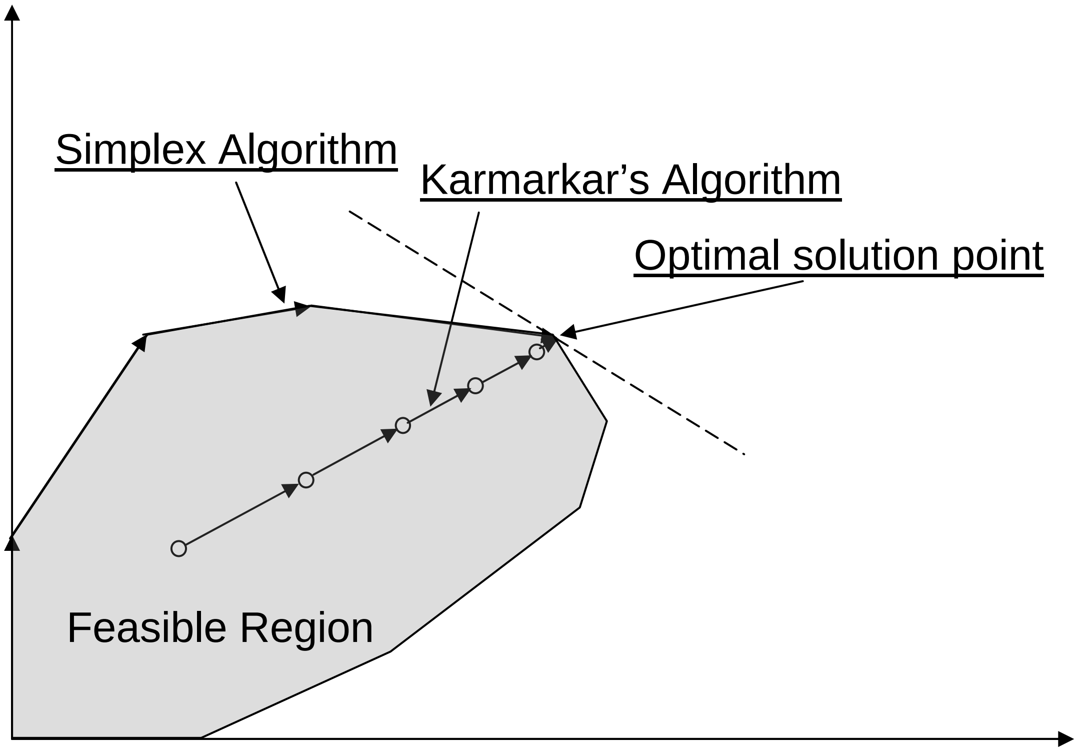
Few other methods, for solving LP problems, use an entirely different algorithmic philosophy.

- *Khatchian's ellipsoid method*
- *Karmarkar's projective scaling method*

Objectives

- To present a comparative discussion between new methods and Simplex method
- To discuss in detail about Karmarkar's projective scaling method

Comparative discussion between new methods and Simplex method



Khatchian's ellipsoid method and Karmarkar's projective scaling method seek the optimum solution to an LP problem by moving through the interior of the feasible region.

Comparative discussion between new methods and Simplex method

1. Both *Khatchian's ellipsoid method* and *Karmarkar's projective scaling method* have been shown to be polynomial time algorithms.

   Time required for an LP problem of size n is at most an^b , where a and b are two positive numbers.

2. Simplex algorithm is an exponential time algorithm in solving LP problems.

   Time required for an LP problem of size n is at most $c2^n$, where c is a positive number

Comparative discussion between new methods and Simplex method

3. For a large enough n (with positive a, b and c), $c2^n > an^b$.

   The polynomial time algorithms are computationally superior to exponential algorithms for large LP problems.

4. However, the rigorous computational effort of *Karmarkar's projective scaling method,* is not economical for 'not-so-large' problems.

Karmarkar’s projective scaling method

- Also known as *Karmarkar’s interior point LP algorithm*
- Starts with a trial solution and shoots it towards the optimum solution
- LP problems should be expressed in a particular form

Karmarkar's projective scaling method

LP problems should be expressed in the following form:

$$\begin{aligned} \text{Minimize} \quad & Z = \mathbf{C}^{\mathrm{T}}\mathbf{X} \\ \text{subject to:} \quad & \mathbf{AX} = \mathbf{0} \\ & \mathbf{1X} = 1 \\ \text{with:} \quad & \mathbf{X} \geq \mathbf{0} \end{aligned}$$

where

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad \mathbf{1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}_{(1\times n)} \quad \mathbf{A} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

and $n \geq 2$

Karmarkar's projective scaling method

It is also assumed that $\mathbf{X}_0 = \begin{bmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{bmatrix}$ is a feasible solution and $Z_{\min} = 0$

Two other variables are defined as: $r = \dfrac{1}{\sqrt{n(n-1)}}$ and $\alpha = \dfrac{(n-1)}{3n}$.

Karmarkar's projective scaling method follows iterative steps to find the optimal solution

Karmarkar's projective scaling method

In general, k^{th} iteration involves following computations

a) Compute $\mathbf{C}_p = \left[\mathbf{I} - \mathbf{P}^{\mathrm{T}}\left(\mathbf{P}\mathbf{P}^{\mathrm{T}}\right)^{-1}\mathbf{P}\right]\overline{\mathbf{C}}^{\mathrm{T}}$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{A}\mathbf{D}_k \\ \mathbf{1} \end{pmatrix} \qquad \overline{\mathbf{C}} = \mathbf{C}^{\mathrm{T}}\mathbf{D}_k \qquad \mathbf{D}_k = \begin{bmatrix} \mathbf{X}_k(1) & 0 & 0 & 0 \\ 0 & \mathbf{X}_k(2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mathbf{X}_k(n) \end{bmatrix}$$

If , $\mathbf{C}_p = \mathbf{0}$, any feasible solution becomes an optimal solution. Further iteration is not required. Otherwise, go to next step.

Karmarkar's projective scaling method

b) $$\mathbf{Y}_{new} = \mathbf{X}_0 - \alpha\, r \frac{\mathbf{C}_p}{\left\|\mathbf{C}_p\right\|}$$

c) $$\mathbf{X}_{k+1} = \frac{\mathbf{D}_k \mathbf{Y}_{new}}{\mathbf{1D}_k \mathbf{Y}_{new}}$$

However, it can be shown that for k =0, $\dfrac{\mathbf{D}_k \mathbf{Y}_{new}}{\mathbf{1D}_k \mathbf{Y}_{new}} = \mathbf{Y}_{new}$ Thus, $\mathbf{X}_1 = \mathbf{Y}_{new}$

d) $$Z = \mathbf{C}^{\mathrm{T}} \mathbf{X}_{k+1}$$

e) Repeat the steps (a) through (d) by changing k as k+1

Karmarkar's projective scaling method: Example

Consider the LP problem:

$$\begin{aligned} &\text{Minimize} && Z = 2x_2 - x_3 \\ &\text{subject to:} && x_1 - 2x_2 + x_3 = 0 \\ &&& x_1 + x_2 + x_3 = 1 \\ &&& x_1, x_2, x_3 \geq 0 \end{aligned}$$

Thus, $n = 3 \quad \mathbf{C} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \quad \mathbf{X}_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ \vdots \\ 1/3 \end{bmatrix}$

and also, $r = \dfrac{1}{\sqrt{n(n-1)}} = \dfrac{1}{\sqrt{3(3-1)}} = \dfrac{1}{\sqrt{6}} \qquad \alpha = \dfrac{(n-1)}{3n} = \dfrac{(3-1)}{3 \times 3} = \dfrac{2}{9}$

Karmarkar's projective scaling method: Example

Iteration 0 (k=0):

$$\mathbf{D}_0 = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\overline{\mathbf{C}} = \mathbf{C}^T \mathbf{D}_0 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 0 & 2/3 & -1/3 \end{bmatrix}$$

$$\mathbf{A}\mathbf{D}_0 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 & 1/3 \end{bmatrix}$$

Karmarkar's projective scaling method: Example

Iteration 0 (k=0)…contd.:

$$\mathbf{P} = \begin{pmatrix} \mathbf{AD}_0 \\ \mathbf{1} \end{pmatrix} = \begin{bmatrix} 1/3 & -2/3 & 1/3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{PP}^{\mathrm{T}} = \begin{bmatrix} 1/3 & -2/3 & 1/3 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1/3 & 1 \\ -2/3 & 1 \\ 1/3 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\left(\mathbf{PP}^{\mathrm{T}}\right)^{-1} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1/3 \end{bmatrix}$$

Karmarkar's projective scaling method: Example

Iteration 0 (k=0)…contd.:

$$\mathbf{P}^{\mathrm{T}}\left(\mathbf{P}\mathbf{P}^{\mathrm{T}}\right)^{-1}\mathbf{P} = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

$$\mathbf{C}_p = \left[\mathbf{I} - \mathbf{P}^{\mathrm{T}}\left(\mathbf{P}\mathbf{P}^{\mathrm{T}}\right)^{-1}\mathbf{P}\right]\overline{\mathbf{C}}^{\mathrm{T}} = \begin{bmatrix} 1/6 \\ 0 \\ -1/6 \end{bmatrix}$$

$$\left\|\mathbf{C}_p\right\| = \sqrt{(1/6)^2 + 0 + (1/6)^2} = \frac{\sqrt{2}}{6}$$

Karmarkar's projective scaling method: Example

Iteration 0 (k=0)…contd.:

$$\mathbf{Y}_{new} = \mathbf{X}_0 - \alpha r \frac{\mathbf{C}_p}{\left\|\mathbf{C}_p\right\|} = \begin{bmatrix} 1/3 \\ 1/3 \\ \vdots \\ 1/3 \end{bmatrix} - \frac{\frac{2}{9} \times \frac{1}{\sqrt{6}}}{\frac{\sqrt{2}}{6}} \times \begin{bmatrix} 1/6 \\ 0 \\ -1/6 \end{bmatrix} = \begin{bmatrix} 0.2692 \\ 0.3333 \\ 0.3974 \end{bmatrix}$$

$$\mathbf{X}_1 = \mathbf{Y}_{new} = \begin{bmatrix} 0.2692 \\ 0.3333 \\ 0.3974 \end{bmatrix} \qquad Z = \mathbf{C}^T \mathbf{X}_1 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} 0.2692 \\ 0.3333 \\ 0.3974 \end{bmatrix} = 0.2692$$

Karmarkar's projective scaling method: Example

Iteration 1 (k=1):

$$\mathbf{D}_1 = \begin{bmatrix} 0.2692 & 0 & 0 \\ 0 & 0.3333 & 0 \\ 0 & 0 & 0.3974 \end{bmatrix}$$

$$\bar{\mathbf{C}} = \mathbf{C}^{\mathrm{T}}\mathbf{D}_1 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} 0.2692 & 0 & 0 \\ 0 & 0.3333 & 0 \\ 0 & 0 & 0.3974 \end{bmatrix} = \begin{bmatrix} 0 & 0.6667 & -0.3974 \end{bmatrix}$$

$$\mathbf{AD}_1 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 0.2692 & 0 & 0 \\ 0 & 0.3333 & 0 \\ 0 & 0 & 0.3974 \end{bmatrix} = \begin{bmatrix} 0.2692 & -0.6666 & 0.3974 \end{bmatrix}$$

Karmarkar's projective scaling method: Example

Iteration 1 (k=1)…contd.:

$$\mathbf{P} = \begin{pmatrix} \mathbf{AD}_1 \\ \mathbf{1} \end{pmatrix} = \begin{bmatrix} 0.2692 & -0.6667 & 0.3974 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{PP}^{\mathrm{T}} = \begin{bmatrix} 0.2692 & -0.6667 & 0.3974 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0.2692 & 1 \\ -0.6667 & 1 \\ 0.3974 & 1 \end{bmatrix} = \begin{bmatrix} 0.675 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\mathbf{P}^{\mathrm{T}} \left(\mathbf{PP}^{\mathrm{T}} \right)^{-1} \mathbf{P} = \begin{bmatrix} 0.441 & 0.067 & 0.492 \\ 0.067 & 0.992 & -0.059 \\ 0.492 & -0.059 & 0.567 \end{bmatrix}$$

Karmarkar's projective scaling method: Example

Iteration 1 (k=1)…contd.:

$$\mathbf{C}_p = \left[\mathbf{I} - \mathbf{P}^{\mathrm{T}}\left(\mathbf{P}\mathbf{P}^{\mathrm{T}}\right)^{-1}\mathbf{P}\right]\bar{\mathbf{C}}^{\mathrm{T}} = \begin{bmatrix} 0.151 \\ -0.018 \\ -0.132 \end{bmatrix}$$

$$\left\|\mathbf{C}_p\right\| = \sqrt{(0.151)^2 + (-0.018)^2 + (-0.132)^2} = 0.2014$$

$$\mathbf{Y}_{new} = \mathbf{X}_0 - \alpha\, r \frac{\mathbf{C}_p}{\left\|\mathbf{C}_p\right\|} = \begin{bmatrix} 1/3 \\ 1/3 \\ \vdots \\ 1/3 \end{bmatrix} - \frac{\frac{2}{9} \times \frac{1}{\sqrt{6}}}{0.2014} \times \begin{bmatrix} 0.151 \\ -0.018 \\ -0.132 \end{bmatrix} = \begin{bmatrix} 0.2653 \\ 0.3414 \\ 0.3928 \end{bmatrix}$$

Karmarkar's projective scaling method: Example

Iteration 1 (k=1)…contd.:

$$\mathbf{D}_1\mathbf{Y}_{new} = \begin{bmatrix} 0.2692 & 0 & 0 \\ 0 & 0.3333 & 0 \\ 0 & 0 & 0.3974 \end{bmatrix} \times \begin{bmatrix} 0.2653 \\ 0.3414 \\ 0.3928 \end{bmatrix} = \begin{bmatrix} 0.0714 \\ 0.1138 \\ 0.1561 \end{bmatrix}$$

$$\mathbf{1D}_1\mathbf{Y}_{new} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0.0714 \\ 0.1138 \\ 0.1561 \end{bmatrix} = 0.3413$$

Karmarkar's projective scaling method: Example

Iteration 1 (k=1)…contd.:

$$\mathbf{X}_2 = \frac{\mathbf{D}_1\mathbf{Y}_{new}}{\mathbf{1D}_1\mathbf{Y}_{new}} = \frac{1}{0.3413} \times \begin{bmatrix} 0.0714 \\ 0.1138 \\ 0.1561 \end{bmatrix} = \begin{bmatrix} 0.2092 \\ 0.3334 \\ 0.4574 \end{bmatrix}$$

$$Z = \mathbf{C}^{\mathrm{T}}\mathbf{X}_2 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} 0.2092 \\ 0.3334 \\ 0.4574 \end{bmatrix} = 0.2094$$

Two successive iterations are shown. Similar iterations can be followed to get the final solution upto some predefined tolerance level

Thank You