Linear Programming

Revised Simplex Method, Duality of LP problems and Sensitivity analysis

Introduction

Revised simplex method is an improvement over *simplex method*. It is computationally more efficient and accurate.

Duality of LP problem is a useful property that makes the problem easier in some cases

Dual simplex method is computationally similar to simplex method. However, their approaches are different from each other.

Primal-Dual relationship is also helpful in *sensitivity or post optimality analysis* of decision variables.

Objectives

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- To explain *revised simplex method*
- To discuss about *duality of LP* and Primal-Dual relationship
- To illustrate *dual simplex method*
- To end with *sensitivity or post optimality analysis*

Revised Simplex method: Introduction

- Benefit of revised simplex method is clearly comprehended in case of large LP problems.
- In simplex method the entire simplex tableau is updated while a small part of it is used.
- The revised simplex method uses exactly the same steps as those in simplex method.
- The only difference occurs in the details of computing the entering variables and departing variable.

Revised Simplex method

Consider the following LP problem (with general notations, after transforming it to its standard form and incorporating all required slack, surplus and artificial variables)

$$
\begin{array}{llllllllllll}
(Z) & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \cdots\cdots\cdots & + & c_nx_n + Z & = 0 \\
(x_i) & c_{11}x_1 & + & c_{12}x_2 & + & c_{13}x_3 & + & \cdots\cdots\cdots & + & c_{1n}x_n & = b_1 \\
(x_j) & c_{21}x_1 & + & c_{22}x_2 & + & c_{23}x_3 & + & \cdots\cdots\cdots & + & c_{2n}x_n & = b_2 \\
\vdots & \vdots & & & & & & & & & \vdots \\
\vdots & \vdots & & & & & & & & & \vdots \\
(x_l) & c_{m1}x_1 & + & c_{m2}x_2 & + & c_{m3}x_3 & + & \cdots\cdots\cdots & + & c_{mn}x_n & = b_m
\end{array}
$$

As the revised simplex method is mostly beneficial for large LP problems, it will be discussed in the context of matrix notation.

Revised Simplex method: Matrix form

Matrix notation

$$\text{Minimize } \; z = \mathbf{C}^{\mathrm{T}}\mathbf{X}$$
$$\text{subject to: } \; \mathbf{AX} = \mathbf{B}$$
$$\text{with: } \mathbf{X} \geq \mathbf{0}$$

where

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

Revised Simplex method: Notations

Notations for subsequent discussions:

Column vector corresponding to a decision variable x_k is $\begin{bmatrix} c_{1k} \\ c_{2k} \\ \vdots \\ c_{mk} \end{bmatrix}$.

$\mathbf{X_S}$ is the column vector of basic variables

$\mathbf{C_S}$ is the row vector of cost coefficients corresponding to $\mathbf{X_S}$, and

$\mathbf{S}$ is the basis matrix corresponding to $\mathbf{X_S}$

Revised Simplex method: Iterative steps

1. Selection of entering variable

For each of the nonbasic variables, calculate the coefficient ($WP - c$), where, P is the corresponding column vector associated with the nonbasic variable at hand, c is the cost coefficient associated with that nonbasic variable and $W = C_S S^{-1}$.

For maximization (minimization) problem, nonbasic variable, having the lowest negative (highest positive) coefficient, as calculated above, is the entering variable.

Revised Simplex method: Iterative steps

2. Selection of departing variable
   a) A new column vector **U** is calculated as **U** = $\mathbf{S}^{-1}$ **B**
   b) Corresponding to the entering variable, another vector **V** is calculated as **V** = $\mathbf{S}^{-1}$ **P**, where **P** is the column vector corresponding to entering variable.
   c) It may be noted that length of both **U** and **V** is same (= m). For $i = 1, \ldots, m$, the ratios, **U**(i)/**V**(i), are calculated provided **V**(i) > 0. $i = r$, for which the ratio is least, is noted. The r th basic variable of the current basis is the departing variable.

      If it is found that **V**(i) < 0 for all i, then further calculation is stopped concluding that bounded solution does not exist for the LP problem at hand.

Revised Simplex method: Iterative steps

3. <u>Update to new Basis</u>

Old basis **S**, is updated to new basis $\mathbf{S}_{\mathbf{new}}$, as $\mathbf{S}_{\mathbf{new}} = [\,\mathbf{E}\ \mathbf{S}^{-1}\,]^{-1}$

where

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & \cdots & \eta_1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \eta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \eta_r & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \eta_{m-1} & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \eta_m & \cdots & 0 & 1 \end{bmatrix} \quad \text{and} \quad \eta_i = \begin{cases} \dfrac{V(i)}{V(r)} & \text{for} \quad i \neq r \\ \dfrac{1}{V(r)} & \text{for} \quad i = r \end{cases}$$

r thcolumn

Revised Simplex method: Iterative steps

S is replaced by $\mathbf{S}_{\mathbf{new}}$ and steps1 through 3 are repeated.

If all the coefficients calculated in step 1, i.e., is positive (negative) in case of maximization (minimization) problem, then optimum solution is reached

The optimal solution is

$$\mathbf{X}_S = \mathbf{S}^{-1}\mathbf{B} \quad \text{and} \quad z = \mathbf{C}\mathbf{X}_S$$

Duality of LP problems

- Each LP problem (called as Primal in this context) is associated with its counterpart known as Dual LP problem.
- Instead of primal, solving the dual LP problem is sometimes easier in following cases

a) The dual has fewer constraints than primal

Time required for solving LP problems is directly affected by the number of constraints, i.e., number of iterations necessary to converge to an optimum solution, which in Simplex method usually ranges from 1.5 to 3 times the number of structural constraints in the problem

b) The dual involves maximization of an objective function

It may be possible to avoid artificial variables that otherwise would be used in a primal minimization problem.

Finding Dual of a LP problem

Primal	**Dual**
Maximization	Minimization
Minimization	Maximization
i^{th} variable	i^{th} constraint
j^{th} constraint	j^{th} variable
$x_i > 0$	Inequality sign of i^{th} Constraint: $\geq$ if dual is maximization $\leq$ if dual is minimization

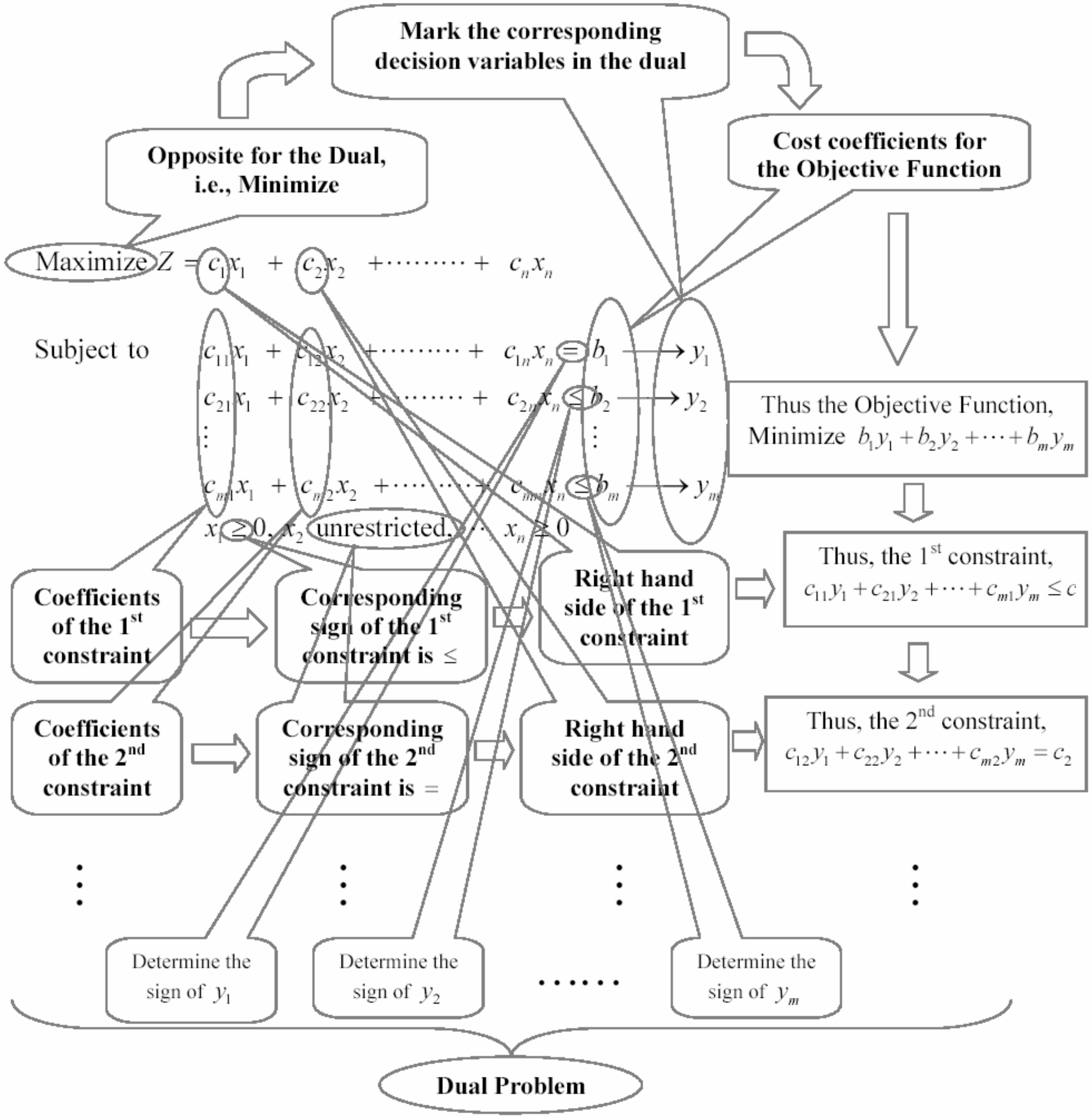
…contd. to next slide

Finding Dual of a LP problem…contd.

Primal	Dual
i^{th} variable unrestricted	i^{th} constraint with = sign
j^{th} constraint with = sign	j^{th} variable unrestricted
RHS of j^{th} constraint	Cost coefficient associated with j^{th} variable in the objective function
Cost coefficient associated with i^{th} variable in the objective function	RHS of i^{th} constraint constraints

Refer class notes for pictorial representation of all the operations

Dual from a Primal



Mark the corresponding decision variables in the dual

Opposite for the Dual, i.e., Minimize

Cost coefficients for the Objective Function

Maximize $Z = c_1x_1 + c_2x_2 + \cdots\cdots\cdots + c_nx_n$

Subject to

$c_{11}x_1 + c_{12}x_2 + \cdots\cdots\cdots + c_{1n}x_n = b_1 \longrightarrow y_1$

$c_{21}x_1 + c_{22}x_2 + \cdots\cdots\cdots + c_{2n}x_n \le b_2 \longrightarrow y_2$

$\vdots$

$c_{m1}x_1 + c_{m2}x_2 + \cdots\cdots\cdots + c_{mn}x_n \le b_m \longrightarrow y_m$

$x_1 \ge 0,\ x_2$ unrestricted, $\cdots,\ x_n \ge 0$

Thus the Objective Function, Minimize $b_1y_1 + b_2y_2 + \cdots + b_my_m$

Coefficients of the 1st constraint

Corresponding sign of the 1st constraint is $\le$

Right hand side of the 1st constraint

Thus, the 1st constraint, $c_{11}y_1 + c_{21}y_2 + \cdots + c_{m1}y_m \le c$

Coefficients of the 2nd constraint

Corresponding sign of the 2nd constraint is $=$

Right hand side of the 2nd constraint

Thus, the 2nd constraint, $c_{12}y_1 + c_{22}y_2 + \cdots + c_{m2}y_m = c_2$

⋮

Determine the sign of y_1

Determine the sign of y_2

……

Determine the sign of y_m

Dual Problem

Dual Problem

Minimize $Z = b_1y_1 + b_2y_2 + \cdots\cdots\cdots + b_my_m$

Subject to

$$c_{11}y_1 + c_{21}y_2 + \cdots\cdots\cdots + c_{m1}y_m \le c_1$$

$$c_{12}y_1 + c_{22}y_2 + \cdots\cdots\cdots + c_{m2}y_m = c_2$$

$$\vdots$$

$$c_{1n}y_1 + c_{2n}y_2 + \cdots\cdots\cdots + c_{mn}y_m \le c_n$$

y_1 unrestricted, $y_2 \ge 0,\ \cdots,\ y_m \ge 0$

Finding Dual of a LP problem…contd.

Note:

Before finding its dual, all the constraints should be transformed to ‘less-than-equal-to’ or ‘equal-to’ type for maximization problem and to ‘greater-than-equal-to’ or ‘equal-to’ type for minimization problem.

It can be done by multiplying with -1 both sides of the constraints, so that inequality sign gets reversed.

Finding Dual of a LP problem: An example

Primal	Dual
Maximize $Z = 4x_1 + 3x_2$	Minimize $Z' = 6000y_1 - 2000y_2 + 4000y_3$
Subject to $x_1 + \frac{2}{3}x_2 \le 6000$ $x_1 - x_2 \ge 2000$ $x_1 \le 4000$ x_1 unrestricted $x_2 \ge 0$	Subject to $y_1 - y_2 + y_3 = 4$ $\frac{2}{3}y_1 + y_2 \le 3$ $y_1 \ge 0$ $y_2 \ge 0$ $y_3 \ge 0$

Note: Second constraint in the primal is transformed to $-x_1 + x_2 \le -2000$ before constructing the dual.

Primal-Dual relationships

- If one problem (either primal or dual) has an optimal feasible solution, other problem also has an optimal feasible solution. The optimal objective function value is same for both primal and dual.
- If one problem has no solution (infeasible), the other problem is either infeasible or unbounded.
- If one problem is unbounded the other problem is infeasible.

Dual Simplex Method

Simplex Method verses Dual Simplex Method

1. Simplex method starts with a nonoptimal but feasible solution where as dual simplex method starts with an optimal but infeasible solution.

2. Simplex method maintains the feasibility during successive iterations where as dual simplex method maintains the optimality.

Dual Simplex Method: Iterative steps

Steps involved in the dual simplex method are:

1. All the constraints (except those with equality (=) sign) are modified to 'less-than-equal-to' sign. Constraints with greater-than-equal-to' sign are multiplied by -1 through out so that inequality sign gets reversed. Finally, all these constraints are transformed to equality sign by introducing required slack variables.
2. Modified problem, as in step one, is expressed in the form of a simplex tableau. If all the cost coefficients are positive (i.e., optimality condition is satisfied) and one or more basic variables have negative values (i.e., non-feasible solution), then dual simplex method is applicable.

Dual Simplex Method: Iterative steps…contd.

3. **Selection of exiting variable:** The basic variable with the highest negative value is the exiting variable. If there are two candidates for exiting variable, any one is selected. The row of the selected exiting variable is marked as pivotal row.
4. **Selection of entering variable:** Cost coefficients, corresponding to all the negative elements of the pivotal row, are identified. Their ratios are calculated after changing the sign of the elements of pivotal row, i.e.,

$$ratio = \left(\frac{Cost\ Coefficients}{-1 \times Elements\ of\ \ pivotal\ row} \right)$$

The column corresponding to minimum ratio is identified as the pivotal column and associated decision variable is the entering variable.

Dual Simplex Method: Iterative steps…contd.

5. **Pivotal operation:** Pivotal operation is exactly same as in the case of simplex method, considering the pivotal element as the element at the intersection of pivotal row and pivotal column.

6. **Check for optimality:** If all the basic variables have nonnegative values then the optimum solution is reached. Otherwise, Steps 3 to 5 are repeated until the optimum is reached.

Dual Simplex Method: An Example

Consider the following problem:

$$\begin{aligned} \text{Minimize} \quad & Z = 2x_1 + x_2 \\ \text{subject to} \quad & x_1 \geq 2 \\ & 3x_1 + 4x_2 \leq 24 \\ & 4x_1 + 3x_2 \geq 12 \\ & -x_1 + 2x_2 \geq 1 \end{aligned}$$

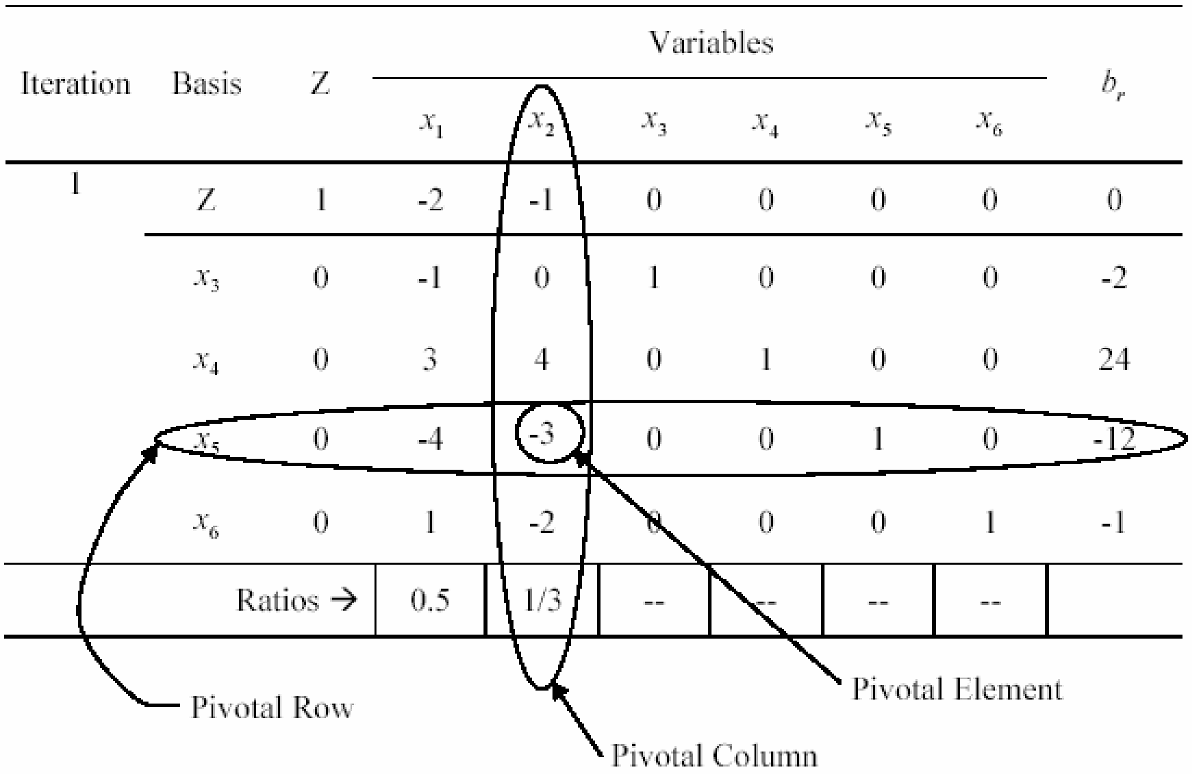
Dual Simplex Method: An Example…contd.

After introducing the surplus variables the problem is reformulated with equality constraints as follows:

$$\begin{array}{llllll}
\text{Minimize} & Z = 2x_1 + x_2 & & & & \\
\text{subject to} & -x_1 & & +x_3 & = & -2 \\
 & 3x_1 & +4x_2 & +x_4 & = & 24 \\
 & -4x_1 & -3x_2 & +x_5 & = & -12 \\
 & x_1 & -2x_2 & +x_6 & = & -1
\end{array}$$

Dual Simplex Method: An Example…contd.

Expressing the problem in the tableau form:



Iteration	Basis	Z	Variables						b_r
			x_1	x_2	x_3	x_4	x_5	x_6	
1	Z	1	-2	-1	0	0	0	0	0
	x_3	0	-1	0	1	0	0	0	-2
	x_4	0	3	4	0	1	0	0	24
	x_5	0	-4	-3	0	0	1	0	-12
	x_6	0	1	-2	0	0	0	1	-1
	Ratios →		0.5	1/3	--	--	--	--	

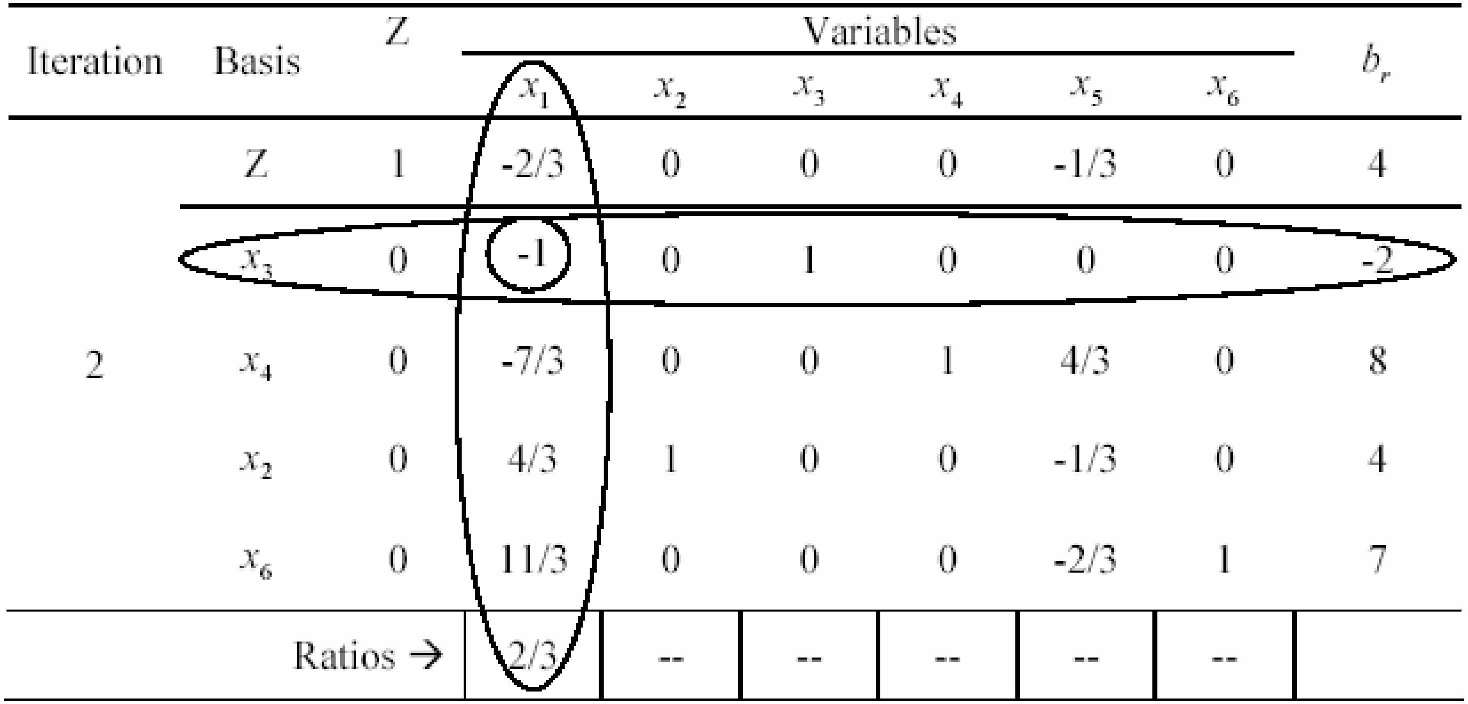
Pivotal Row

Pivotal Element

Pivotal Column

Dual Simplex Method: An Example…contd.

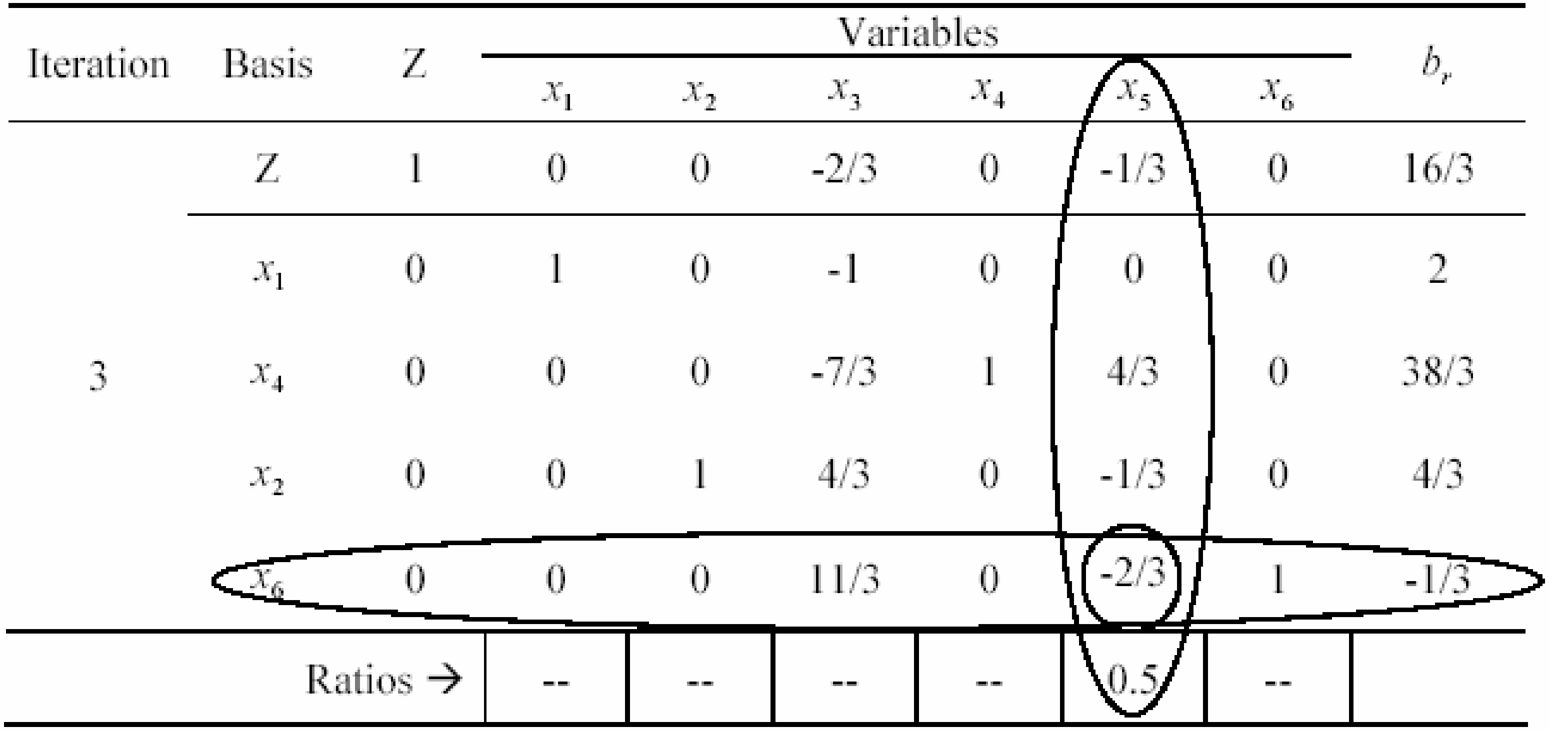
Successive iterations:



Iteration	Basis	Z	Variables						b_r
			x_1	x_2	x_3	x_4	x_5	x_6	
2	Z	1	-2/3	0	0	0	-1/3	0	4
	x_3	0	-1	0	1	0	0	0	-2
	x_4	0	-7/3	0	0	1	4/3	0	8
	x_2	0	4/3	1	0	0	-1/3	0	4
	x_6	0	11/3	0	0	0	-2/3	1	7
	Ratios →		2/3	--	--	--	--	--	

Dual Simplex Method: An Example…contd.

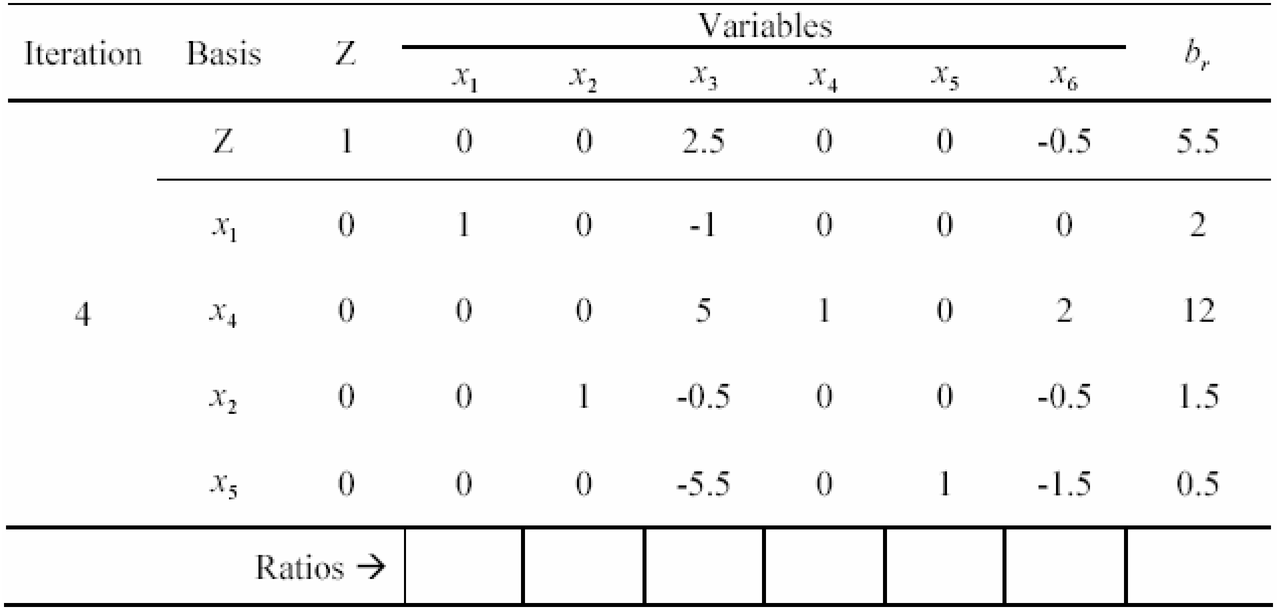
Successive iterations:



Iteration	Basis	Z	Variables						b_r
			x_1	x_2	x_3	x_4	x_5	x_6	
3	Z	1	0	0	-2/3	0	-1/3	0	16/3
	x_1	0	1	0	-1	0	0	0	2
	x_4	0	0	0	-7/3	1	4/3	0	38/3
	x_2	0	0	1	4/3	0	-1/3	0	4/3
	x_6	0	0	0	11/3	0	-2/3	1	-1/3
	Ratios →		--	--	--	--	0.5	--	

Dual Simplex Method: An Example…contd.

Successive iterations:



Iteration	Basis	Z	Variables						b_r
			x_1	x_2	x_3	x_4	x_5	x_6	
4	Z	1	0	0	2.5	0	0	-0.5	5.5
	x_1	0	1	0	-1	0	0	0	2
	x_4	0	0	0	5	1	0	2	12
	x_2	0	0	1	-0.5	0	0	-0.5	1.5
	x_5	0	0	0	-5.5	0	1	-1.5	0.5
		Ratios →							

As all the b_r are positive, optimum solution is reached.
Thus, the optimal solution is Z = 5.5 with x_1 = 2 and x_2 = 1.5

Solution of Dual from Primal Simplex

Primal

$$\text{Maximize} \quad Z = 4x_1 - x_2 + 2x_3$$

$$\text{subject to} \quad 2x_1 + x_2 + 2x_3 \le 6$$

$$x_1 - 4x_2 + 2x_3 \le 0$$

$$5x_1 - 2x_2 - 2x_3 \le 4$$

$$x_1, x_2, x_3 \ge 0$$

Dual

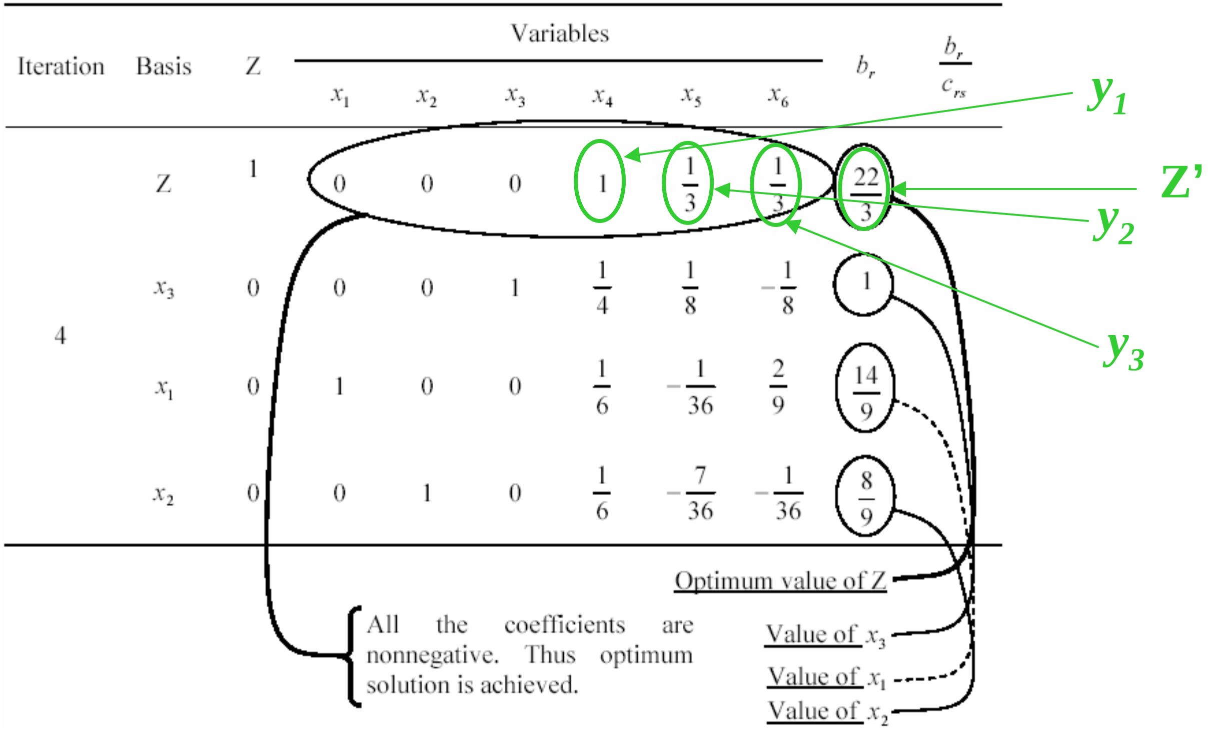
$$\text{Minimize} \quad Z' = 6y_1 + 0y_2 + 4y_3$$

$$\text{subject to} \quad 2y_1 + y_2 + 5y_3 \ge 4$$

$$y_1 - 4y_2 - 2y_3 \ge -1$$

$$2y_1 + 2y_2 - 2y_3 \ge 2$$

$$y_1, y_2, y_3 \ge 0$$



Iteration	Basis	Z	Variables x_1	x_2	x_3	x_4	x_5	x_6	b_r	$\frac{b_r}{c_{rs}}$
4	Z	1	0	0	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{22}{3}$	
	x_3	0	0	0	1	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{1}{8}$	1	
	x_1	0	1	0	0	$\frac{1}{6}$	$-\frac{1}{36}$	$\frac{2}{9}$	$\frac{14}{9}$	
	x_2	0	0	1	0	$\frac{1}{6}$	$-\frac{7}{36}$	$-\frac{1}{36}$	$\frac{8}{9}$	

Optimum value of Z

All the coefficients are nonnegative. Thus optimum solution is achieved.

Value of x_3

Value of x_1

Value of x_2

Sensitivity or post optimality analysis

- Changes that can affect only Optimality
  - Change in coefficients of the objective function, C_1, C_2,..
  - Re-solve the problem to obtain the solution
- Changes that can affect only Feasibility
  - Change in right hand side values, b_1, b_2,..
  - Apply dual simplex method or study the dual variable values
- Changes that can affect both Optimality and Feasibility
  - Simultaneous change in C_1, C_2,.. and b_1, b_2,..
  - Use both primal simplex and dual simplex or re-solve

Sensitivity or post optimality analysis

A dual variable, associated with a constraint, indicates a change in *Z* value (optimum) for a small change in RHS of that constraint.

$$\Delta Z = y_j \Delta b_i$$

where y_j is the dual variable associated with the i^{th} constraint,

Δb_i is the small change in the RHS of i^{th} constraint,

ΔZ is the change in objective function owing to Δb_i.

Sensitivity or post optimality anal An Example

Let, for a LP problem, ith constraint be

$$2x_1 + x_2 \le 50$$

and the optimum value of the objective function be 250.

RHS of the i^{th} constraint changes to 55, i.e., i^{th} constraint changes to

$$2x_1 + x_2 \le 55$$

Let, dual variable associated with the i^{th} constraint is y_j, optimum value of which is 2.5 (say). Thus, Δb_i = 55 – 50 = 5 and y_j = 2.5

So, $\Delta Z = y_j \, \Delta b_i$ = 2.5x5 = 12.5 and revised optimum value of the objective function is 250 + 12.5 = 262.5.

Thank You