Linear Programming

Preliminaries

Objectives

- To introduce linear programming problems (LPP)
- To discuss the standard and canonical form of LPP
- To discuss elementary operation for linear set of equations

Introduction and Definition

- *Linear Programming* (LP) is the most useful optimization technique
- Objective function and constraints are the ‘linear’ functions of ‘nonnegative’ decision variables
- Thus, the conditions of LP problems are
    1. Objective function must be a linear function of decision variables
    2. Constraints should be linear function of decision variables
    3. All the decision variables must be nonnegative

Example

$$\begin{aligned}
&\text{Maximize} && Z = 6x + 5y && \prec \text{Objective Function}\\
&\text{subject to} && 2x - 3y \le 5 && \prec \text{1st Constraint}\\
& && x + 3y \le 11 && \prec \text{2nd Constraint}\\
& && 4x + y \le 15 && \prec \text{3rd Constraint}\\
& && x,\ y \ge 0 && \prec \text{Nonnegativity Condition}
\end{aligned}$$

This is in “general” form

Standard form of LP problems

- Standard form of LP problems must have following three characteristics:

    1. Objective function should be of maximization type
    2. All the constraints should be of equality type
    3. All the decision variables should be nonnegative

General form Vs Standard form

- General form

$$\begin{aligned} \text{Minimize} \quad & Z = -3x_1 - 5x_2 \\ \text{subject to} \quad & 2x_1 - 3x_2 \le 15 \\ & x_1 + x_2 \le 3 \\ & 4x_1 + x_2 \ge 2 \\ & x_1 \ge 0 \\ & x_2 \text{ unrestricted} \end{aligned}$$

- Violating points for standard form of LPP:

1. Objective function is of minimization type
2. Constraints are of inequality type
3. Decision variable, x_2, is unrestricted, thus, may take negative values also.

How to transform a general form of a LPP to the standard form ?

General form —Transformation→ Standard form

- General form
  1. Objective function

     Minimize $Z = -3x_1 - 5x_2$

  2. First constraint

     $2x_1 - 3x_2 \le 15$

  3. Second constraint

     $x_1 + x_2 \le 3$

- Standard form
  1. Objective function

     Maximize $Z' = -Z = 3x_1 + 5x_2$

  2. First constraint

     $2x_1 - 3x_2 + x_3 = 15$

  3. Second constraint

     $x_1 + x_2 + x_4 = 3$

Variables x_3 and x_4 are known as slack variables

General form $\xrightarrow{\text{Transformation}}$ Standard form

- General form

4. Third constraint

$$4x_1 + x_2 \geq 2$$

- Standard form

4. Third constraint

$$4x_1 + x_2 - x_5 = 2$$

Variable x_5 is known as surplus variable

- General form

5. Constraints for decision variables, x_1 and x_2

$$x_1 \geq 0$$
$$x_2 \text{ unrestricted}$$

- Standard form

5. Constraints for decision variables, x_1 and x_2

$$x_1 \geq 0$$
$$x_2 = x_2' - x_2''$$
$$\text{and } \; x_2', \; x_2'' \geq 0$$

Canonical form of LP Problems

- The ‘objective function’ and all the ‘equality constraints’ (standard form of LP problems) can be expressed in *canonical form*.
- This is known as canonical form of LPP
- *Canonical form* of LP problems is essential for *simplex method* (will be discussed later)
- *Canonical form* of a set of linear equations will be discussed next.

Canonical form of a set of linear equations

Let us consider the following example of a set of linear equations

$$3x + 2y + z = 10 \quad (A_0)$$

$$x - 2y + 3z = 6 \quad (B_0)$$

$$2x + y - z = 1 \quad (C_0)$$

The system of equation will be transformed through '*Elementary Operations*'.

Elementary Operations

The following operations are known as *elementary operations*:

1. Any equation E_r can be replaced by kE_r, where k is a nonzero constant.
2. Any equation E_r can be replaced by $E_r + kE_s$, where E_s is another equation of the system and k is as defined above.

Note: Transformed set of equations through *elementary operations* is equivalent to the original set of equations. Thus, solution of transformed set of equations is the solution of original set of equations too.

Transformation to Canonical form: An Example

Set of equation (A_0, B_0 and C_0) is transformed through *elementary operations* (shown inside bracket in the right side)

$$x + \frac{2}{3}y + \frac{1}{3}z = \frac{10}{3} \qquad \left(A_1 = \frac{1}{3}A_0\right)$$

$$0 - \frac{8}{3}y + \frac{8}{3}z = \frac{8}{3} \qquad \left(B_1 = B_0 - A_1\right)$$

$$0 - \frac{1}{3}y - \frac{5}{3}z = -\frac{17}{3} \qquad \left(C_1 = C_0 - 2A_1\right)$$

Note that variable *x* is eliminated from B_0 and C_0 equations to obtain B_1 and C_1. Equation A_0 is known as pivotal equation.

Transformation to Canonical form: Example contd.

Following similar procedure, y is eliminated from equation A_1 and C_1 considering B_1 as pivotal equation:

$$x + 0 + z = 4 \qquad \left(A_2 = A_1 - \frac{2}{3} B_2 \right)$$

$$0 + y - z = -1 \qquad \left(B_2 = -\frac{3}{8} B_1 \right)$$

$$0 + 0 - 2z = -6 \qquad \left(C_2 = C_1 + \frac{1}{3} B_2 \right)$$

Transformation to Canonical form: Example contd.

Finally, z is eliminated form equation A_2 and B_2 considering C_2 as pivotal equation :

$$x + 0 + 0 = 1 \qquad (A_3 = A_2 - C_3)$$

$$0 + y + 0 = 2 \qquad (B_3 = B_2 + C_3)$$

$$0 + 0 + z = 3 \qquad \left(C_3 = -\frac{1}{2} C_2\right)$$

Note: Pivotal equation is transformed first and using the transformed pivotal equation other equations in the system are transformed.

The set of equations (A_3, B_3 and C_3) is said to be in ***Canonical form*** which is equivalent to the original set of equations (A_0, B_0 and C_0)

Pivotal Operation

Operation at each step to eliminate one variable at a time, from all equations except one, is known as ***pivotal operation***.

Number of *pivotal operations* are same as the number of variables in the set of equations.

Three *pivotal operations* were carried out to obtain the canonical form of set of equations in last example having three variables.

Transformation to Canonical form: Generalized procedure

Consider the following system of n equations with n variables

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots + a_{1n}x_n = b_1 \qquad (E_1)$$

$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots\cdots + a_{2n}x_n = b_2 \qquad (E_2)$$

$$\vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots\cdots + a_{nn}x_n = b_n \qquad (E_n)$$

Transformation to Canonical form: Generalized procedure

- *Canonical form* of above system of equations can be obtained by performing *n* pivotal operations
- Variable $x_i \; (i = 1 \cdots n)$ is eliminated from all equations except j^{th} equation for which a_{ji} is nonzero.
- General procedure for one pivotal operation consists of following two steps,
  1. Divide j^{th} equation by a_{ji}. Let us designate it as (E'_j), i.e., $E'_j = \dfrac{E_j}{a_{ji}}$
  2. Subtract a_{ki} times of (E'_j) equation from k^{th} equation $\left(k = 1, 2, \cdots j-1, j+1, \cdots, n\right)$, i.e., $E_k - a_{ki}E'_j$

Transformation to Canonical form: Generalized procedure

After repeating above steps for all the variables in the system of equations, the canonical form will be obtained as follows:

$$1x_1 + 0x_2 + \cdots\cdots\cdots + 0x_n = b_1'' \qquad (E_1^c)$$

$$0x_1 + 1x_2 + \cdots\cdots\cdots + 0x_n = b_2'' \qquad (E_2^c)$$

$$\vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots$$

$$0x_1 + 0x_2 + \cdots\cdots\cdots + 1x_n = b_n'' \qquad (E_n^c)$$

It is obvious that solution of above set of equation such as $x_i = b_i''$ is the solution of original set of equations also.

Transformation to Canonical form: More general case

Consider more general case for which the system of equations has m equation with n variables $(n \geq m)$

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots + a_{1n}x_n = b_1 \qquad (E_1)$$

$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots\cdots + a_{2n}x_n = b_2 \qquad (E_2)$$

$$\vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots\cdots + a_{mn}x_n = b_m \qquad (E_m)$$

It is possible to transform the set of equations to an equivalent canonical form from which at least one solution can be easily deduced

Transformation to Canonical form: More general case

By performing *n pivotal operations* for any *m* variables (say, $x_1, x_2, \cdots x_m$, called *pivotal variables*) the system of equations reduced to *canonical form* is as follows

$$1x_1 + 0x_2 + \cdots\cdots\cdots + 0x_m + a''_{1,m+1}x_{m+1} + \cdots\cdots\cdots + a''_{1n}x_n = b''_1 \qquad (E_1^c)$$

$$0x_1 + 1x_2 + \cdots\cdots\cdots + 0x_m + a''_{2,m+1}x_{m+1} + \cdots\cdots\cdots + a''_{2n}x_n = b''_2 \qquad (E_2^c)$$

$$\vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots$$

$$0x_1 + 0x_2 + \cdots\cdots\cdots + 1x_m + a''_{m,m+1}x_{m+1} + \cdots\cdots\cdots + a''_{mn}x_n = b''_m \qquad (E_m^c)$$

Variables, $x_{m+1}, \cdots, x_n$, of above set of equations is known as *nonpivotal variables* or independent variables.

Basic variable, Nonbasic variable, Basic solution, Basic feasible solution

One solution that can be obtained from the above set of equations is

$$x_i = b_i'' \quad \text{for} \quad i = 1, \cdots, m$$
$$x_i = 0 \quad \text{for} \quad i = (m+1), \cdots, n$$

This solution is known as *basic solution*.

Pivotal variables, $x_1, x_2, \cdots x_m$, are also known as ***basic variables***.

Nonpivotal variables, $x_{m+1}, \cdots, x_n$, are known as ***nonbasic variables***.

Basic solution is also known as ***basic feasible solution*** because it satisfies all the constraints as well as nonnegativity criterion for all the variables

Thank You