Module 4: Frequency Domain Spectral Analysis

Exercise Problems

4.8. For the exercise problem 3.13 (Chapter 3), compare between the plots of the PSDFs of the relative displacement x obtained for the two cases of excitation. Use the correlation function as $\rho_{ij} = \exp\left(-\frac{r_{ij}\omega}{2\pi V_s}\right)$ (Equation 2.93) and the PSDF of ground acceleration as given in the Appendix, same for the two supports. Assume $\frac{r}{V_s} =$ 5s for illustrative purpose only. Also, compare between the rms values of the response.

4.9. A suspended span of a submarine pipeline is modeled as shown in Figure 4.20. Assuming 5% modal damping, obtain a closed form expression for the PSDF of the relative displacement x. Assume the PSDF of ground acceleration to be the same at the two supports and represented by a white noise with constant PSDF S_0. The cross spectral density function between supports 1 and 2 is given by $S_{g_1g_2} = S_0 e^{-i\omega T}$ and the equivalent stiffness corresponding to the d. o. f. x is k with $\frac{k}{m} = 100(\text{rad/s})^2$.

Figure 4.20

4.10. For the exercise problem 3.14, obtain plots of $S_{yy}(\omega)$ and $S_{y\theta}(\omega)$ (both real and imaginary components) assuming (i) the same PSDF of excitations at all supports (given in the Appendix) (ii) the same PSDF but partially correlated at the supports. For the latter, assume the same correlation function as given in problem 4.8 having a time lag of 2.5s between supports.

4.11. For the exercise problem 3.16, find the rms values of the top relative displacement and the drift between the first and the second story. Assume the time lag between the supports as 2.5s. Take the correlation function and the PSDF same as those given in exercise problem 4.8.

4.12. For the exercise problem 3.17, find the rms values of the absolute displacement of the secondary system and the base shear for perfectly correlated support excitations represented by the PSDF given in the Appendix.

4.13. Using the modal spectral analysis, find the peak values of the displacement (relative) of the top floor and the first story drift of the frame of exercise problem 3.18 for perfectly correlated ground excitations represented by the PSDF given in the Appendix.

4.14. Using the modal and state space spectral analyses, find the rms value of the deflection of the centre of the deck for the exercise problem 3.19. Compare the results for the two cases (i) perfectly correlated excitations at the supports and (ii) partially correlated excitations with a time lag between supports as 2.5s. Take the PSDF of ground excitation (horizontal) as that given in the Appendix and use the correlation function used in the problem 4.8.

4.15. For the exercise problem 3.21, find the rms values of displacements and rotations of the top floor of the 3D tall building using modal spectral analysis. Also, obtain a plot of $S_{x\theta}$ (both real and imaginary components). Take the excitation as the ground motion, represented by the PSDF given in the appendix, applied in the *x* direction.

4.16. For the shear frame shown in the exercise problem 3.20, compare between the rms values of absolute accelerations of the top floor and the bending moment at the base obtained by the direct, modal and state space spectral analyses. Take the same PSDF of excitation which is used for other problems.

Take the relevant figures from the slides or from the reference book

Module 4: Frequency Domain Spectral Analysis

Exercise Solution :

<u>ERRATA FOR THE TEXT BOOK</u>

pp 178, 4th para, 8th line: $\phi \neq 225^0$, but 180^0

pp 179, 2nd line of Equation 4.29 is redundant

pp 186, Equation 4.76 should be $S_{\ddot{x}_g x} = -HMIS_{\ddot{x}_g}$

pp 189, Example 4.3: Because of the use of wrong r matrix (Example 3.9) PSDF of DOF (5) for fully correlated excitation is non zero. For correct r and K matrices, the rms responses are

DOF(4) = 0.0237m	DOF(5) = 0.00081m	partially corrected
DOF(4) = 0.0332m	DOF(5)=0	fully corrected

The shape of the PSDFs remain the same as shown in Figure 4.13 and 4.14(b)

pp 197, Example 4.6: Because of the use of wrong r matrix (Example 3.11), the values of the ordinates of Figures 4.17 and 4.18 will change, but the shape remains the same. For the correct value of r matrix, rms values of displacement of DOF (1), left tower, and DOF(3, not 2 as printed), centre of the deck, are 0.0219m and 0.0152m respectively.

Refer to the exercise problem 4.8 and Figure 3.22

From the exercise problem 3.13., the effective stiffness and effective mass corresponding to the d.o.f x are taken as

$$K_x = 341.8k \quad M = 2.33m \quad \frac{k}{m} = 200$$

$$\omega_n = \sqrt{\frac{K_x}{M}} = 12.118\, rad/\sec$$

$$\left|H(\omega)\right|^2 = \left[\left(\omega_n^2 - \omega^2\right)^2 + 4\eta^2\omega^2\omega_n^2\right]^{-2} \qquad \xi = 5\%$$

$$S_x(\omega) = \left|H(\omega)\right|^2 S_{\ddot{x}g}(\omega)$$

The PSDF of the displacement x is shown in Figure 4.21. The rms value of x is the square root of the area under the PSDF curve.

rms of the displacement x [from PSDF curve] = 0.011763m rms of the displacement x [from the time history analysis – Ch.3; (problem 3.13)] = 0.012136m

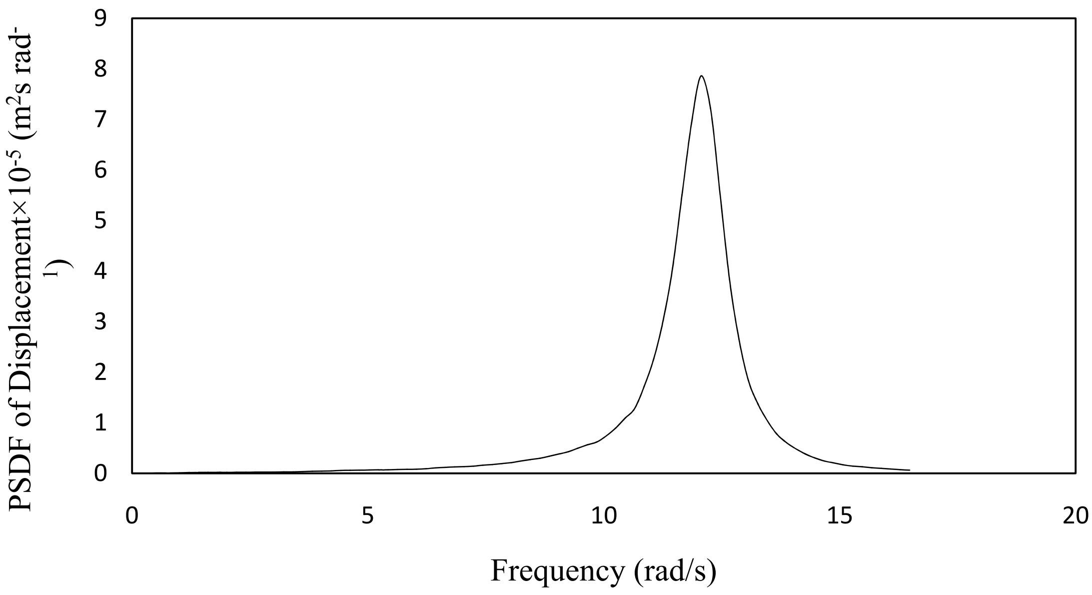


Figure 4.21 PSDF of the x – displacement

Refer to the exercise problem 4.9 and Figure 4.20

$$\omega_n^2 = \frac{K}{m} = 100$$

Since both supports have different excitations, the acceleration experienced by the mass (Figure 4.20) is given by

$$\ddot{a} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} \ddot{x}_{g1} \\ \ddot{x}_{g2} \end{Bmatrix} = \boldsymbol{A}\ddot{\boldsymbol{x}}_{\boldsymbol{g}}$$

Equation of motion takes the form

$$m\ddot{x} + c\dot{x} + kx = -m\ddot{a} = -m\boldsymbol{A}\ddot{\boldsymbol{x}}_{\boldsymbol{g}}$$

Dividing by m, the equation becomes

$$\ddot{x} + 2\eta\omega_n\dot{x} + \omega_n^2 x = -\boldsymbol{A}\ddot{\boldsymbol{x}}_{\boldsymbol{g}}$$

$$S_x = \left|H(\omega)\right|^2 S_{\ddot{a}}$$

in which $S_{\ddot{a}} = \boldsymbol{A}\boldsymbol{S}_{\ddot{x}_g}\boldsymbol{A}^T = \boldsymbol{A}\begin{bmatrix} 1 & e^{i\omega T} \\ e^{-i\omega T} & 1 \end{bmatrix}\boldsymbol{A}^T S_0 = \frac{1}{2}\left(1+\cos\omega T\right)S_0$

Therefore,

$$S_x = \frac{1}{2}\left[\left(\omega_n^2 - \omega^2\right)^2 + 4\eta^2\omega^2\omega_n^2\right]^{-2}\left(1+\cos\omega T\right)S_0$$

$$=\frac{1}{2}\left[\left(100-\omega^2\right)^2+\omega^2\right]^{-2}\left(1+\cos\omega T\right)S_0$$

Refer to the exercise problems 4.10, 3.14 and Figure 3.23.

Equation of motion takes the form

$$\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{C}\dot{\boldsymbol{x}}+\boldsymbol{K}\boldsymbol{x}=-\boldsymbol{M}\boldsymbol{I}\ddot{x}_g \qquad \boldsymbol{x}=\begin{Bmatrix} y \\ 0 \end{Bmatrix}$$

in which (exercise problem 3.14)

$$\boldsymbol{M}=\begin{bmatrix} 3 & \\ & 24.75 \end{bmatrix} m; \quad \boldsymbol{K}=\begin{bmatrix} 6.7 & -1.65 \\ -1.65 & 54.375 \end{bmatrix} k; \qquad \boldsymbol{C}=\begin{bmatrix} 3.893 & -0.5 \\ -0.5 & 32 \end{bmatrix} m;$$

For multi support excitations,

$$\boldsymbol{I}=\begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{2}{3}L & \frac{2}{L} & -\frac{2}{L} & -\frac{2}{3}L \end{bmatrix}$$

For the same support excitation,

$$I=\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

For partially correlated ground motion,

$$H(\omega)=\left\{\left[\boldsymbol{K}-\boldsymbol{M}\omega^2\right]-\boldsymbol{i}\boldsymbol{C}\omega\right\}^{-1}$$

$$\boldsymbol{S}_{\boldsymbol{xx}}=\boldsymbol{H}(\omega)\boldsymbol{S}_p\boldsymbol{H}(\omega)^{*T}$$

$$\boldsymbol{S}_p=\boldsymbol{M}\boldsymbol{I}\begin{bmatrix} 1 & e^{\frac{-2.5\omega}{2\pi}} & e^{\frac{-.5\omega}{2\pi}} & e^{\frac{-7.5\omega}{2\pi}} \\ e^{\frac{-2.5\omega}{2\pi}} & 1 & e^{\frac{-2.5\omega}{2\pi}} & e^{\frac{-.5\omega}{2\pi}} \\ e^{\frac{-.5\omega}{2\pi}} & e^{\frac{-2.5\omega}{2\pi}} & 1 & e^{\frac{-2.5\omega}{2\pi}} \\ e^{\frac{-7.5\omega}{2\pi}} & e^{\frac{-.5\omega}{2\pi}} & e^{\frac{-2.5\omega}{2\pi}} & 1 \end{bmatrix}\boldsymbol{I}^T\boldsymbol{M}S_{\ddot{x}_g}$$

For perfectly correlated ground motion,

$$\boldsymbol{S}_p=\boldsymbol{M}\boldsymbol{I}\boldsymbol{I}^{\boldsymbol{T}}\boldsymbol{M}^{\boldsymbol{T}}S_{\ddot{x}_g}$$

The plots of $S_{yy}(\omega)$ and $S_{y\theta}(\omega)$ for the cases of partially correlated and perfectly correlated ground motions are shown in Figures 4.22 and 4.23. The rms values of the responses computed from Figure 4.22 show that the rms value of y for partially correlated ground motion is more than that of fully correlated ground motion. The reason for this is attributed to the pipe undergoing rigid body rotation.

rms for partially correlated ground motion = 0.0552m

rms for fully correlated ground motion = 0.04787m

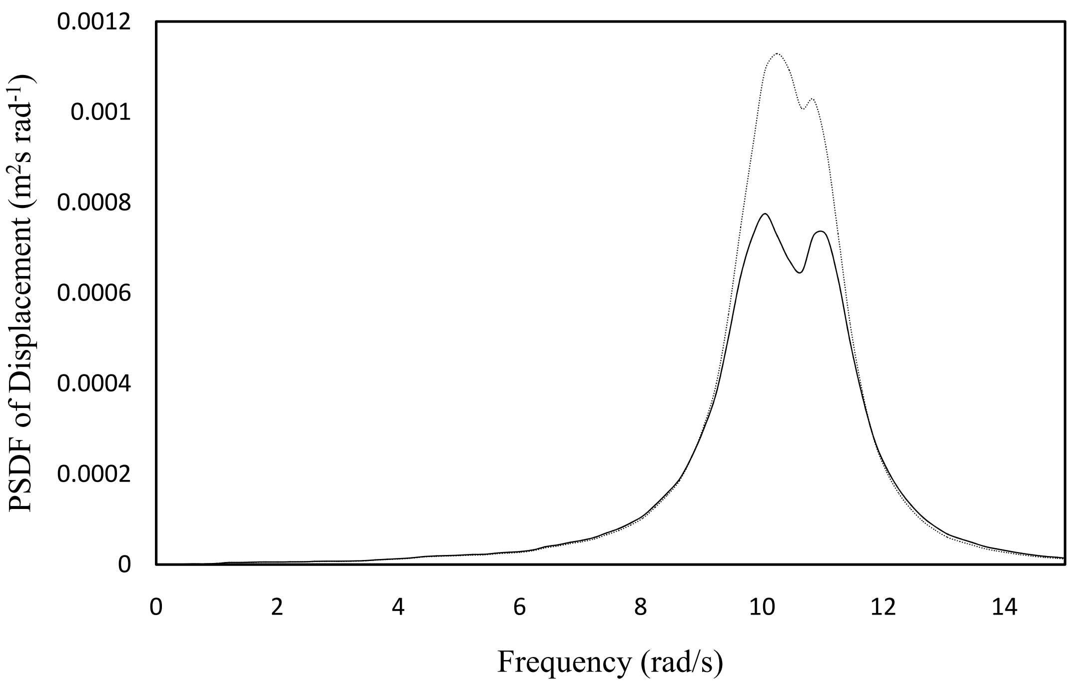


Figure 4.22 PSDF of the y – displacement

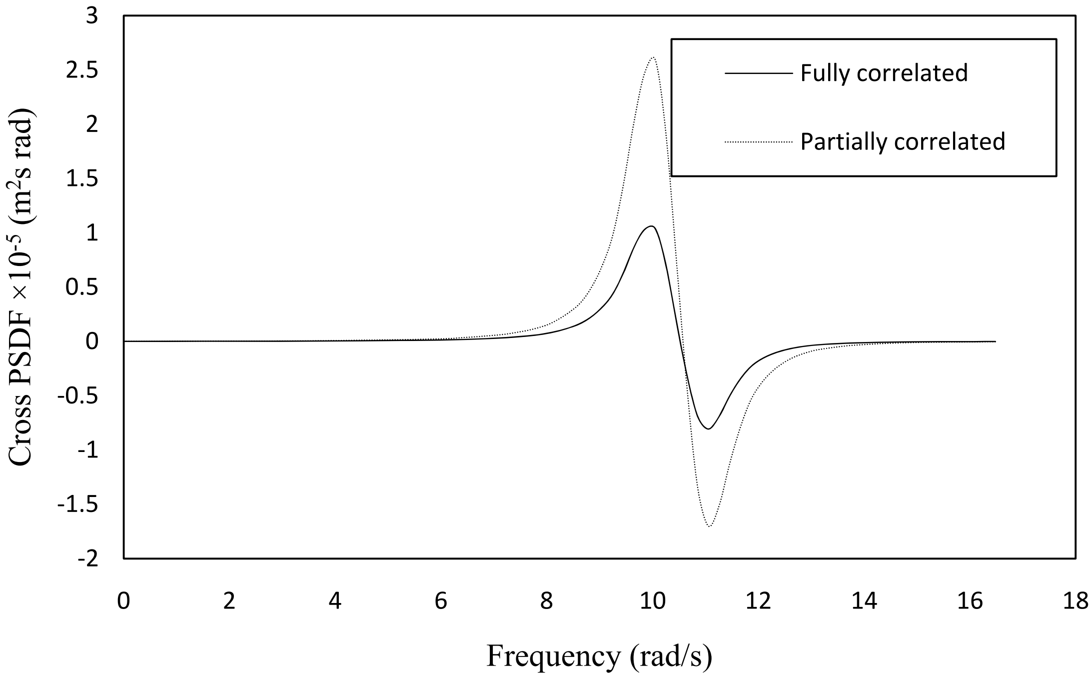


Figure 4.23 Real component of cross PSDF between y and θ

Refer to the exercise problems 4.11, 3.16 and Figure 3.25.

Equation of motion for the frame is given by (exercise problem 3.16)

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = -\boldsymbol{M}\boldsymbol{r}\ddot{\boldsymbol{x}}_{\boldsymbol{g}} \qquad x = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

in which $\boldsymbol{M} = \begin{bmatrix} 2m & \\ & m \end{bmatrix}$; $\quad \boldsymbol{C} = \begin{bmatrix} 2.344 & -0.552 \\ -0.552 & 0.23 \end{bmatrix} m$

$$\boldsymbol{K} = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} k\,; \qquad \boldsymbol{r} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\boldsymbol{S}_x = \boldsymbol{H}(\omega)\,\boldsymbol{S}_p\,\boldsymbol{H}(\omega)^{*T}$$

in which $\boldsymbol{H}(\boldsymbol{\omega}) = \left\{\left[\boldsymbol{K} - \boldsymbol{M}\omega^2\right] + i\boldsymbol{C}\omega\right\}^{-1}$

$$\boldsymbol{S}_{\ddot{x}_g} = \begin{bmatrix} 1 & e^{\frac{-2.5\omega}{2\pi}} & e^{\frac{-.5\omega}{2\pi}} \\ e^{\frac{-2.5\omega}{2\pi}} & 1 & e^{\frac{-2.5\omega}{2\pi}} \\ e^{\frac{-.5\omega}{2\pi}} & e^{\frac{-2.5\omega}{2\pi}} & 1 \end{bmatrix} S_{\ddot{x}_g} \qquad \boldsymbol{S}_p = \boldsymbol{MrS}_{\ddot{x}_g} \boldsymbol{r}^T \boldsymbol{M}^T$$

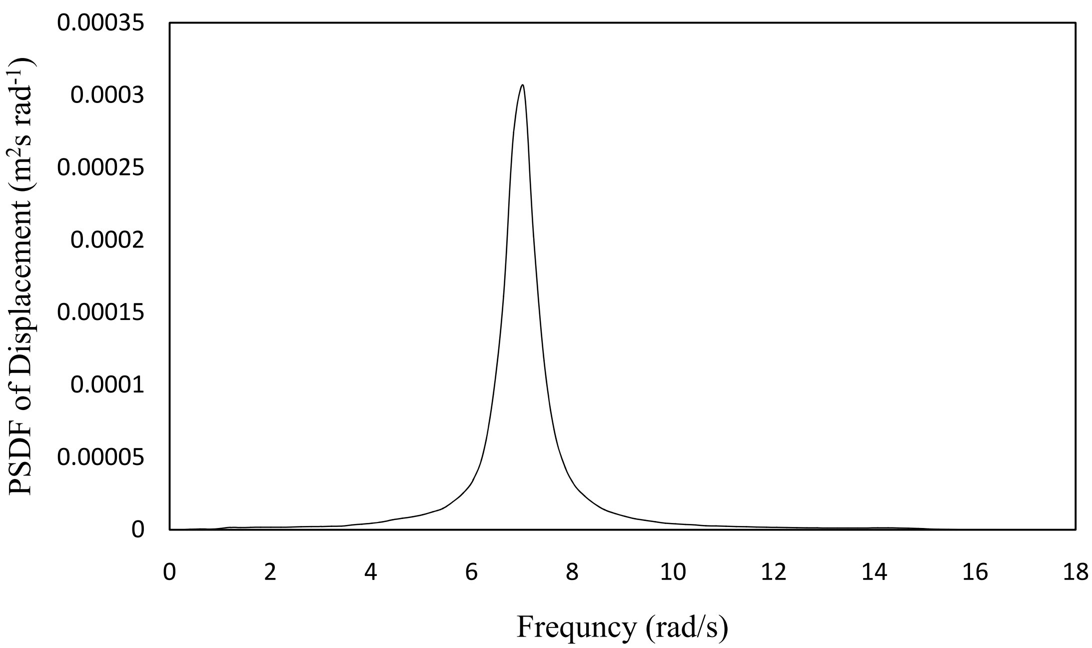
Drift between the second and the first story is $\delta = u_2 - u_1$

$$S_\delta = S_{u_2} + S_{u_1} - S_{u_1 u_2} - S_{u_2 u_1}$$

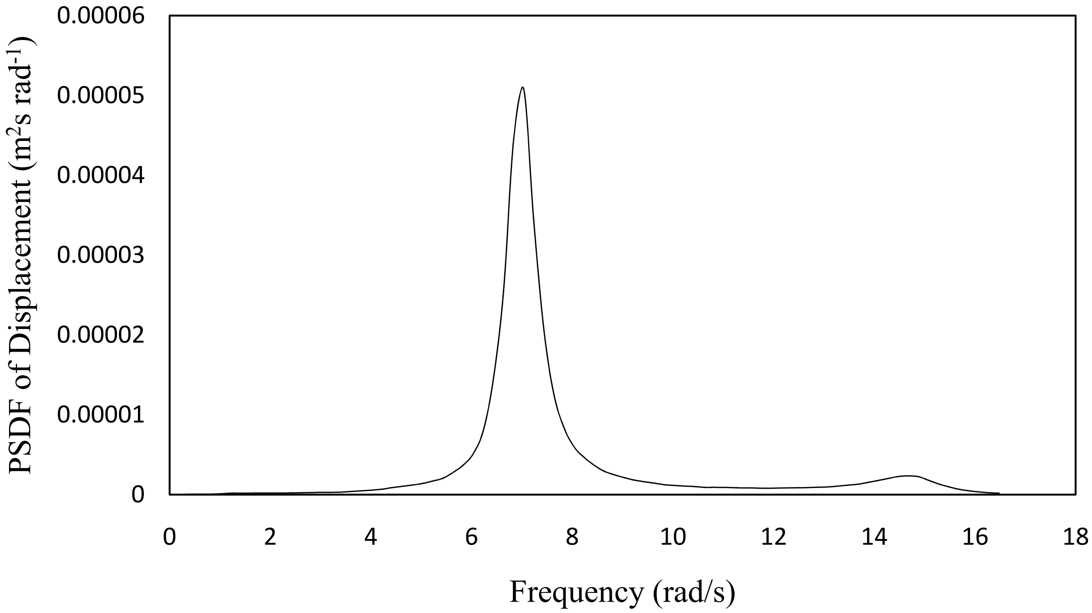
Since $S_{u_2 u_1}$ is complex conjugate of $S_{u_1 u_2}$,

$$S_\delta = S_{u_2} + S_{u_1} - 2\text{real}\left(S_{u_1 u_2}\right)$$

The PSDFs of the S_{u_2} and S_δ are shown in Figure 4.24. The rms values of u_2 and δ are 0.018362m and 0.00786m respectively.



(a)



(b)

Figure 4.24 PSDF of (a) displacement u_2 (b) drift δ

Refer to the exercise problems 4.12, 3.17 and Figure 3.26.

Equations of motion of the primary and the secondary system are given by (the exercise problem 3.17)

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = -\boldsymbol{M}\boldsymbol{I}\ddot{\boldsymbol{x}}_g$$

in which $\boldsymbol{K} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2.187 & -0.187 \\ 0 & -0.187 & 0.187 \end{bmatrix} k$; $\quad \boldsymbol{M} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{4} \end{bmatrix} m$

$$\boldsymbol{C} = \begin{bmatrix} 0.831 & -0.35 & 0 \\ -0.35 & 0.514 & -0.033 \\ 0 & -0.033 & 0.065 \end{bmatrix}$$

$$\boldsymbol{S}_x = \boldsymbol{H}(\boldsymbol{\omega})\boldsymbol{M}\boldsymbol{I}\boldsymbol{I}^T\boldsymbol{M}^T\boldsymbol{H}(\boldsymbol{\omega})^{*T} S_{\ddot{x}_g} \qquad \boldsymbol{I}^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$S_{xx_g} = \frac{1}{\omega^4} H(\omega) MIS_{\dot{x}_g}$$

Absolute-displacement $x_{3a} = (x_3)_{\text{rel}} + x_g \,; = S_{x_3} + S_{x_g} + 2\,\text{real}\left(S_{x_3 x_g}\right)$

The PSDF of the absolute displacement x_{3a} is shown in Figure 4.25. The rms of the absolute displacement x_3 is 0.0413m

The base shear is given by $V = 2kx_1$

$$S_v = 4k^2 S_{x_1}$$

The PSDF of the base shear is shown in Figure 4.26. The rms of the base shear is $2.8437 \times 10^4 k^2$

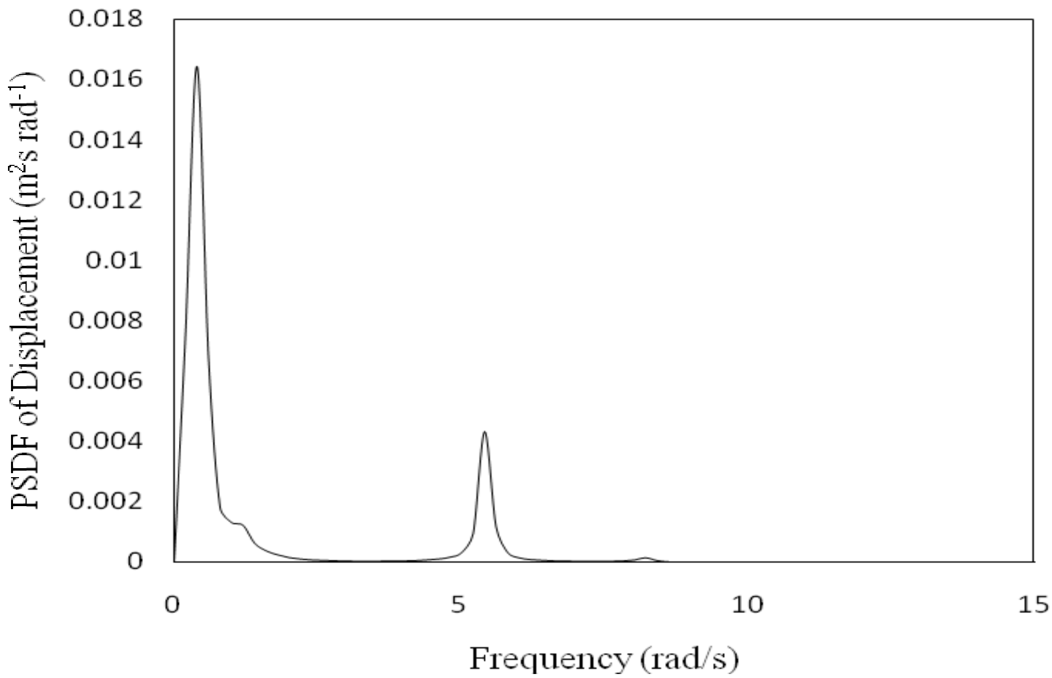


Figure 4.25 PSDF of absolute displacement x_3

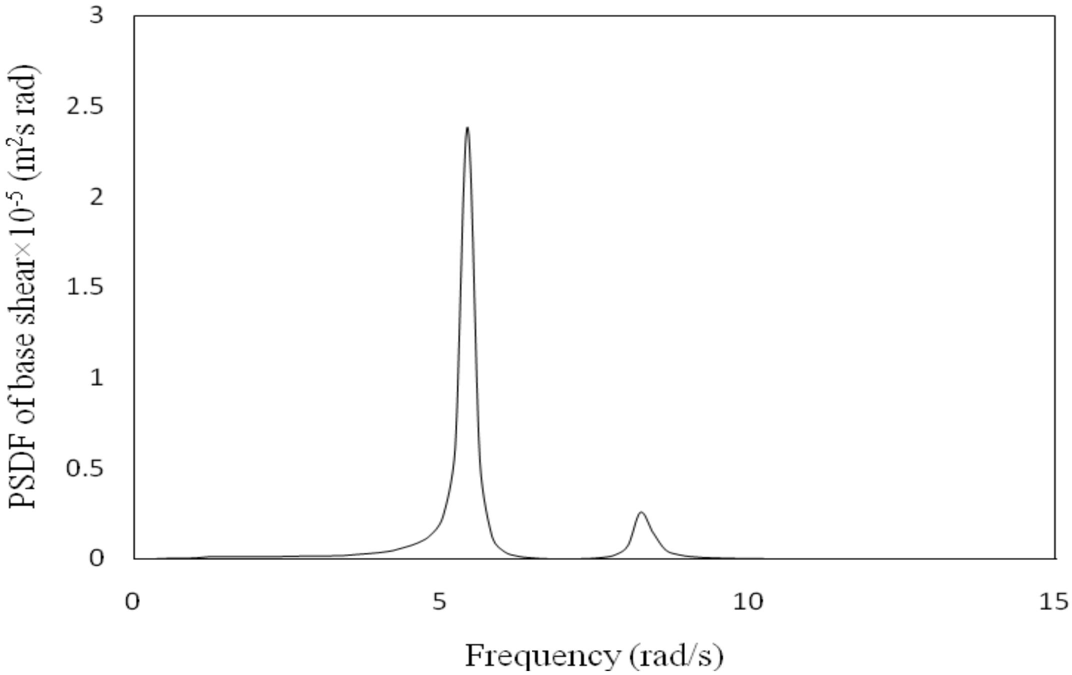


Figure 4.26 PSDF of the base shear

Refer to the exercise problems 4.13, 3.18 and Figure 3.27.

Equations of motion for the frame is (exercise problem 3.18)

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = -\boldsymbol{M}\boldsymbol{I}\ddot{x}_g$$

$$\boldsymbol{K} = \begin{bmatrix} 7 & -3 & 0 & 0 \\ -3 & 5 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} k; \qquad \boldsymbol{M} = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{bmatrix} m$$

Modal load $p_i = -\boldsymbol{\phi}_i^T \boldsymbol{M}\boldsymbol{I}\ddot{x}_g$

$$S_{p_i p_i} = \boldsymbol{\phi}_i^T \boldsymbol{M}\boldsymbol{I}\boldsymbol{I}^T M^T \boldsymbol{\phi}_i S_{\ddot{x}_g}$$

$$S_{p_i p_j} = \boldsymbol{\phi}_i^T \boldsymbol{M}\boldsymbol{I}\boldsymbol{\phi}_j^T \boldsymbol{M}\boldsymbol{I} S_{\ddot{x}_g}$$

$$\omega_1 = 2.54\,\mathrm{rad\,s^{-1}};\ \omega_2 = 5.908\,\mathrm{rad\,s^{-1}};\quad \omega_3 = 9.525\,\mathrm{rad\,s^{-1}};\ \omega_4 = 13.71\,\mathrm{rad\,s^{-1}}$$

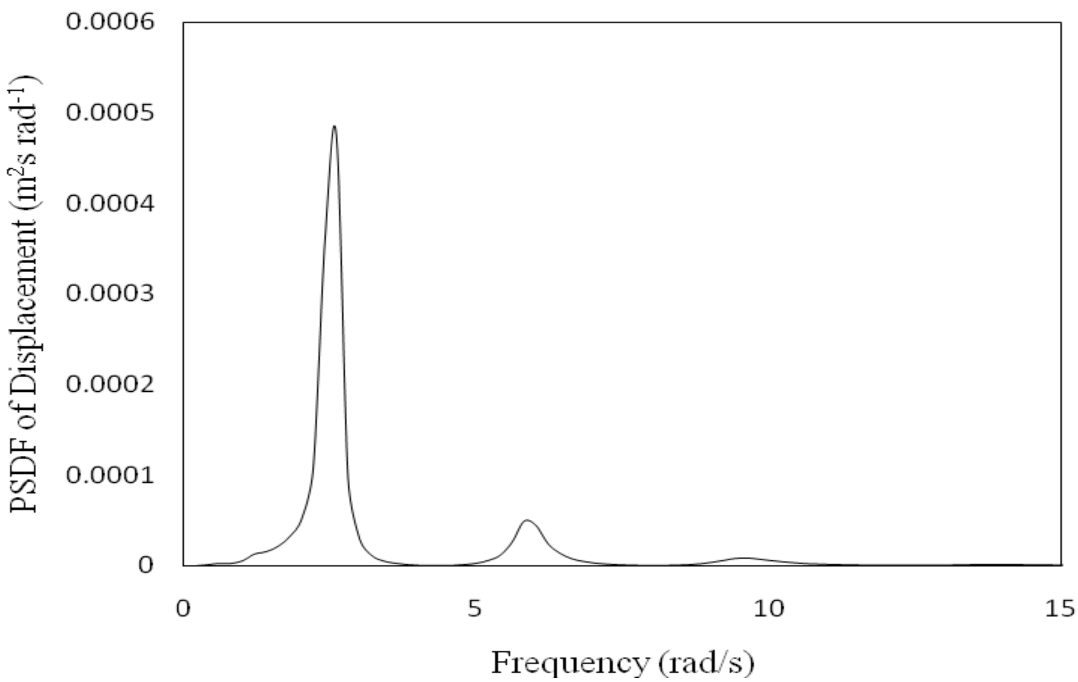
$$S_{z_i} = \left|h_i(\omega)\right|^2 S_{p_i p_i}$$

$$S_{z_i z_j} = h_i(\omega) \overset{*}{h}_j(\omega) S_{p_i p_j}$$

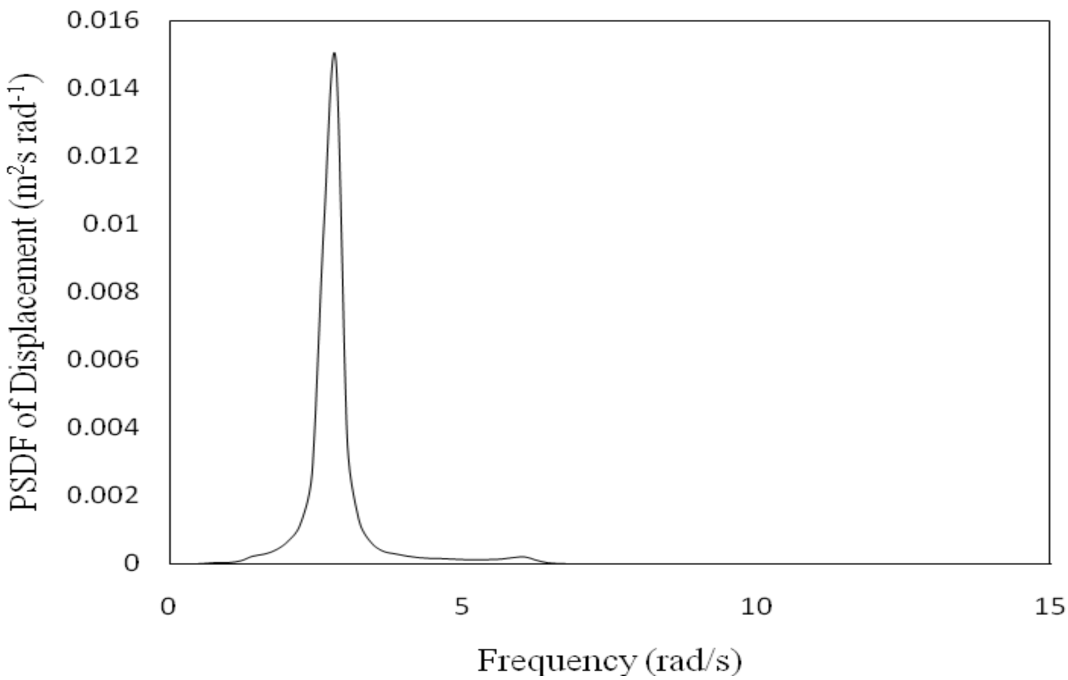
$$h_i(\omega) = \left[\left(\omega_i^2 - \omega^2\right) + 2i\xi\omega_i\right]^{-1}$$

$$\boldsymbol{x} = \boldsymbol{\phi z}\,; \qquad \boldsymbol{S}_{xx} = \boldsymbol{\phi S}_{zz}\boldsymbol{\phi}^T$$

Since the first storey displacement is the same as the first storey drift, the PSDFs of the top storey and the first storey are obtained using the mode shapes of the frame taken from exercise problem 3.18. The PSDFs are shown in Figure 4.27.



(a)



(b)

Figure 4.27 PSDF of (a) top floor displacement (b) first floor displacement

The λ_0, λ_1 and λ_2 for the two PSDFs are calculated and using Equation 2.19c (with T=30s), the peak factors for the top and first storeys are obtained as $p_1 = 2.803$, $p_4 = 2.68$

$\sigma_{x_1} = 0.01748\text{m}$; $\sigma_{x_4} = 0.088\text{m}$; $x_{p_1} = p_1\sigma_{x_1} = 0.049\text{m}$, $x_{p_4} = p_4\sigma_{x_4} = 0.236\text{m}$

Refer to the exercise problems 4.14, 3.19 and Figure 3.28.

The mass, stiffness and damping matrices of the bridge corresponding to the degrees of freedom shown in the exercise problem 3.19

$$\boldsymbol{M} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 60 \end{bmatrix} m \qquad \boldsymbol{C} = \begin{bmatrix} 10.77 & 0 & -1.14 \\ 0 & 10.77 & 1.1845 \\ -1.14 & 1.14 & 18.77 \end{bmatrix} m$$

$$\boldsymbol{K} = \begin{bmatrix} 492 & 0 & -95 \\ 0 & 492 & 95 \\ -95 & 95 & 621.13 \end{bmatrix} m$$

The mode shapes and frequencies are

$$\phi_1 = \begin{bmatrix} -0.31178 & 0.31178 & -1 \end{bmatrix}^T; \qquad \phi_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T \qquad \phi_3 = \begin{bmatrix} -1 & 1 & 0.2078 \end{bmatrix}^T$$

$\omega_1 = 3.06\,\text{rad s}^{-1}$; $\omega_2 = 4.959\,\text{rad s}^{-1}$; $\omega_3 = 5.058\,\text{rad s}^{-1}$; $\alpha = 0.194$; $\beta = 0.0118$

r matrix for multipoint excitation is given as

$$\boldsymbol{r} = \begin{bmatrix} 0.6037 & 0.199 & -0.00606 & -0.01837 \\ -0.0183 & -0.00606 & 0.199149 & 0.6037 \\ 0.095 & 0.03138 & -0.03138 & -0.0951 \end{bmatrix}$$

$$\boldsymbol{S}_p = \boldsymbol{M}\boldsymbol{r}\boldsymbol{S}_{\ddot{x}_g}\boldsymbol{r}^T\boldsymbol{M}^T$$

For fully correlated excitation:

$$\boldsymbol{S}_{\ddot{x}_g} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} S_{\ddot{x}_g}$$

For partially correlated excitation

$$\boldsymbol{S}_{\ddot{x}_g} = \begin{bmatrix} 1 & & & Sym \\ e^{\frac{-2.5\omega}{2\pi}} & 1 & & \\ e^{\frac{-5\omega}{2\pi}} & e^{\frac{-2.5\omega}{2\pi}} & 1 & \\ e^{\frac{-7.5\omega}{2\pi}} & e^{\frac{-5\omega}{2\pi}} & e^{\frac{-2.5\omega}{2\pi}} & 1 \end{bmatrix} S_{\ddot{x}_g}$$

<u>Modal spectral analysis</u>

Using $p_i = \phi_i^T P$; $S_{p_i p_j} = \phi_i^T \boldsymbol{S}_p \phi_j$

$S_{z_i z_j}$ is obtained as in exercise problem 4.13. Finally $\boldsymbol{S}_x$ is determined as

$$\boldsymbol{S}_x = \boldsymbol{\phi}\boldsymbol{S}_z\boldsymbol{\phi}^T$$

<u>State space spectral analysis</u>

For state space spectral analysis, the load vector $\bar{P}$ is given by

$$\bar{P} = \begin{bmatrix} 0 & r\ddot{x}_g \end{bmatrix}^T$$

$\boldsymbol{S}_{\bar{P}}$ matrix is given a $\boldsymbol{S}_{\bar{P}} = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{S}_g \end{bmatrix}$; $\boldsymbol{S}_g = r\boldsymbol{S}_{\ddot{x}_g} r^T$

$$\boldsymbol{S}_x = \boldsymbol{H}(\boldsymbol{\omega})\boldsymbol{S}_{\bar{P}}\boldsymbol{H}(\boldsymbol{\omega})^{*T}$$

$\boldsymbol{H}(\boldsymbol{\omega})$ is obtained as

$$H(\omega) = \begin{bmatrix} \hat{I}\omega & -A \end{bmatrix}^{-1} \qquad A = \begin{bmatrix} 0 & I \\ -KM^{-1} & -CM^{-1} \end{bmatrix}$$

The PSDFs of the centre of the deck displacement obtained by the modal analysis is shown in Figure 4.28. The rms values of the displacement are compared below

	Case (i) (Correlated)	Case (ii) (Partially Correlated)
Modal	0	0.00992 m
State space	0	0.0105 m

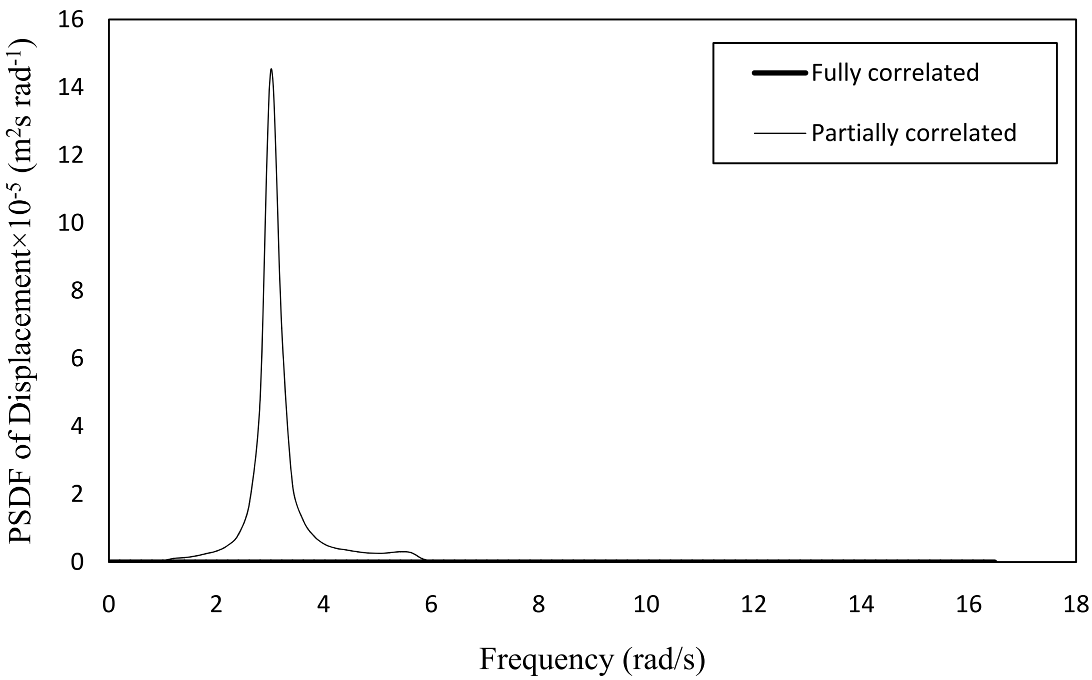


Figure 4.28 PSDF of the centre of the deck displacement

Refer to the exercise problems 4.15, 3.21 and Figure 3.30.

Mass, stiffness and damping matrices for the 3D frame are given as (exercise problem 3.19)

$$K = \begin{bmatrix} 15 & & & & Sym & \\ 0 & 15 & & & & \\ 1.5 & 7.5 & 67.5 & & & \\ -7.5 & 0 & -0.75 & 7.5 & & \\ 0 & -7.5 & -3.75 & 0 & 7.5 & \\ -7.5 & -3.75 & -33.75 & 0.75 & 3.75 & 33.75 \end{bmatrix} k$$

$$\boldsymbol{M} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1.5 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1.5 \end{bmatrix} m \qquad \boldsymbol{I}^T = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The mode shapes and frequencies are

$$\boldsymbol{\phi}_1^T = \begin{bmatrix} 0.12 & 0.618 & -0.1 & 0.2 & 1 & -0.16 \end{bmatrix};\ \omega_1 = 14.5\,\text{rad s}^{-1}$$

$$\boldsymbol{\phi}_2^T = \begin{bmatrix} -0.618 & 0.12 & 0 & -1 & 0.2 & 0 \end{bmatrix};\ \omega_2 = 15.14\,\text{rad s}^{-1}$$

$$\boldsymbol{\phi}_3^T = \begin{bmatrix} -0.03 & -0.148 & -0.618 & -0.048 & -0.24 & -1 \end{bmatrix};\ \omega_3 = 26.58\,\text{rad s}^{-1}$$

$$\boldsymbol{\phi}_4^T = \begin{bmatrix} 0.2 & 1 & -0.166 & -0.124 & -0.618 & 0.103 \end{bmatrix};\ \omega_4 = 38\,\text{rad s}^{-1}$$

$$\boldsymbol{\phi}_5^T = \begin{bmatrix} 1 & -0.2 & 0 & -0.618 & 0.124 & 0 \end{bmatrix};\ \omega_5 = 39.6\,\text{rad s}^{-1}$$

$$\boldsymbol{\phi}_6^T = \begin{bmatrix} -0.05 & -0.24 & -1 & 0.029 & 0.148 & 0.618 \end{bmatrix};\ \omega_6 = 69.6\,\text{rad s}^{-1}$$

Using four modes, the PSDF matrix of the modal load is given as

$$\boldsymbol{S}_{pp} = \boldsymbol{\phi}^T \boldsymbol{M} \boldsymbol{I} \boldsymbol{I}^T \boldsymbol{M}^T \boldsymbol{\phi} S_{\ddot{x}_g}$$

$$\boldsymbol{S}_{pp} = \begin{bmatrix} 1.047 & -5.236 & -0.2513 & 0.247 \\ -5.236 & 2.618 & 1.257 & -1.236 \\ -0.251 & 1.257 & 0.0603 & -0.059 \\ 0.247 & -1.236 & -0.059 & 0.0583 \end{bmatrix} \times 10^7 S_{\ddot{x}_g}$$

Using the same approach as used in the exercise problem 4.13, $S_{z_i z_j}$ is obtained.

$$\boldsymbol{S}_{xx} = \boldsymbol{\phi} \boldsymbol{S}_{zz} \boldsymbol{\phi}^T$$

The plots of S_{xx} and $S_{x\theta}$ for the top floor are shown in Figures 4.29 and 4.30. The rms values of the top floor displacement and rotations are compared below.

Displacement x	8.66×10^{-3} m
Displacement y	9.57×10^{-4} m
Displacement θ	2.77×10^{-4} rad

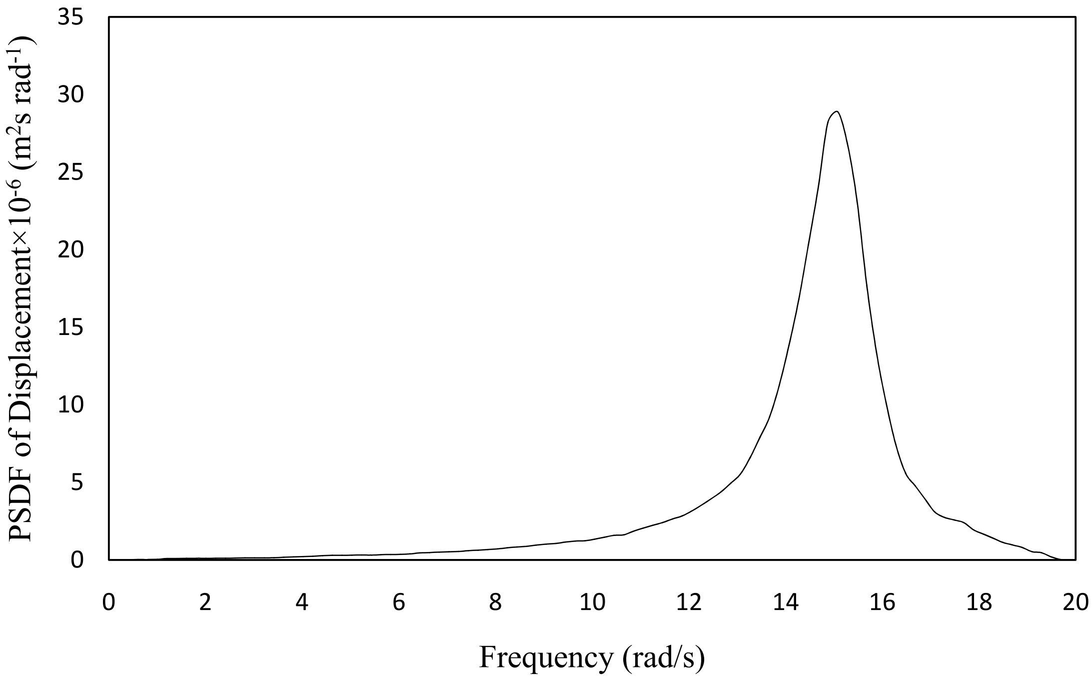


Figure 4.29 PSDF of top floor x – displacement

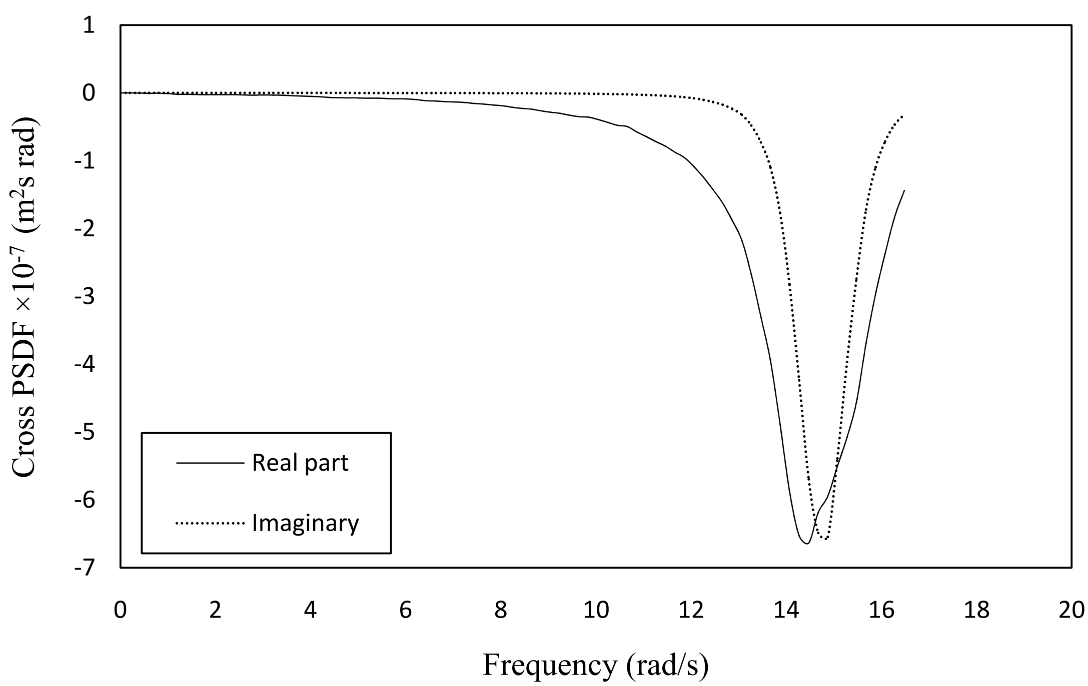


Figure 4.30 Cross PSDF between top floor x- displacement and rotation θ

Refer to the exercise problems 4.16, 3.20 and Figure 3.29.

The mass, stiffness and damping matrices for the frame are (the exercise problem 3.20)

$$\boldsymbol{M}=\begin{bmatrix}1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1\end{bmatrix}m \qquad \boldsymbol{K}=\begin{bmatrix}2 & & & & & Sym \\ -1 & 2 & & & & \\ 0 & -1 & 2 & & & \\ 0 & 0 & -1 & 2 & & \\ 0 & 0 & 0 & -1 & 2 & \\ 0 & 0 & 0 & 0 & -1 & 1\end{bmatrix}k$$

$$\boldsymbol{C}=\begin{bmatrix}1.442 & & & & & \\ -0.664 & 1.442 & & & & \\ 0 & -0.664 & 1.442 & & & \\ 0 & 0 & -0.664 & 1.442 & & \\ 0 & 0 & 0 & -0.664 & 1.442 & \\ 0 & 0 & 0 & 0 & -0.664 & 0.778\end{bmatrix}m$$

$$\boldsymbol{A}=\begin{bmatrix}-80 & 40 & 0 & 0 & 0 & 0 & -1.44 & 0.66 & 0 & 0 & 0 & 0 \\ 40 & -80 & 40 & 0 & 0 & 0 & 0.66 & -1.44 & 0.66 & 0 & 0 & 0 \\ 0 & 40 & -80 & 40 & 0 & 0 & 0 & 0.66 & -1.44 & 0.66 & 0 & 0 \\ 0 & 0 & 40 & -80 & 40 & 0 & 0 & 0 & 0.66 & -1.44 & 0.66 & 0 \\ 0 & 0 & 0 & 40 & -80 & 40 & 0 & 0 & 0 & 0.66 & -1.44 & 0.66 \\ 0 & 0 & 0 & 0 & 40 & -40 & 0 & 0 & 0 & 0 & 0.66 & -0.779\end{bmatrix}$$

Using ω_1 and ω_2,

$\alpha = 0.1138$, $\beta = 0.0166$

$\boldsymbol{\phi}_1^T = \begin{bmatrix}-0.24 & -0.46 & -0.66 & -0.82 & -0.94 & -1\end{bmatrix}$; $\omega_1 = 1.525\,\text{rad s}^{-1}$

$\boldsymbol{\phi}_2^T = \begin{bmatrix}0.66 & 1 & 0.82 & 0.24 & -0.46 & -0.94\end{bmatrix}$; $\omega_2 = 4.485\,\text{rad s}^{-1}$

$\boldsymbol{\phi}_3^T = \begin{bmatrix}-0.94 & -0.66 & 0.461 & 1 & 0.24 & -0.82\end{bmatrix}$; $\omega_3 = 7.185\,\text{rad s}^{-1}$

$\boldsymbol{\phi}_4^T = \begin{bmatrix}1 & -0.24 & -0.94 & 0.46 & 0.82 & -0.66\end{bmatrix}$; $\omega_4 = 9.47\,\text{rad s}^{-1}$

$\boldsymbol{\phi}_5^T = \begin{bmatrix}0.82 & -0.94 & 0.24 & 0.66 & -1 & 0.46\end{bmatrix}$; $\omega_5 = 11.2\,\text{rad s}^{-1}$

$\boldsymbol{\phi}_6^T = \begin{bmatrix}-0.46 & 0.829 & -1 & 0.94 & -0.66 & 0.24\end{bmatrix}$; $\omega_6 = 12.28\,\text{rad s}^{-1}$

The direct, modal and state space spectral analysis are carried out using the methods given in exercise problems 4.11 and 4.16.

Absolute acceleration of the top floor $\ddot{\ddot{x}}_6 = \ddot{x}_6 + \ddot{x}_g$

PSDF of $\ddot{\ddot{x}}_6 = S_{\ddot{x}_6} + S_{\ddot{x}_g} + S_{\ddot{x}_6\ddot{x}_g} + S_{\ddot{x}_g\ddot{x}_6}$

$$\boldsymbol{x}(\omega) = \boldsymbol{H}(\omega)\boldsymbol{MI}\ddot{x}_g(\omega)$$

$$\ddot{\boldsymbol{x}}(\omega) = -\omega^2\boldsymbol{H}(\omega)\boldsymbol{MI}\ddot{x}_g(\omega)$$

$$\boldsymbol{S}_{\ddot{x}_g\ddot{x}} = \omega^4\boldsymbol{H}(\omega)\boldsymbol{MII}^T\boldsymbol{M}^T\boldsymbol{H}(\omega)^* S_{\ddot{x}_g}(\omega)$$

From the matrix of $\boldsymbol{S}_{\ddot{x}_g\ddot{x}_6}$, the term $S_{\ddot{x}_g\ddot{x}_6}$ can be selected. Since $S_{\ddot{x}_6\ddot{x}_g}$ is the complex conjugate of $S_{\ddot{x}_g\ddot{x}_6}$, the expression for PSDF of absolute acceleration $S_{\ddot{\ddot{x}}_6}$ is given by

$$S_{\ddot{\ddot{x}}_6} = S_{\ddot{x}_6} + S_{\ddot{x}_g} + 2real\left(S_{\ddot{x}_6\ddot{x}_g}\right)$$

Bending moment at the base is

$\boldsymbol{M} = kx_1h$ (assuming h=3m)

$$S_M = 9k^2S_{x_1}$$

S_{x_1} can be obtained from the PSDF matrix of the displacement. The rms values of the acceleration and bending moment obtained by different methods are compared below

	Direct	Modal (4 modes)	State space
Absolute acceleration	0.1682 ms^{-2}	0.1592 ms^{-2}	0.1612 ms^{-2}
Base moment	4.2681m	3.94m	4.12m