Module 3 – Response Analysis for Specified Ground Motion

Exercise Problems:

3.13. Find the effective mass and stiffness of the structure corresponding to the dynamic degree of freedom x shown in Figure 3.22. All members are assumed as inextensible. Shear stiffness $\frac{12EI}{l^3}$ for the member AB is k. The shear stiffness of other members are given in terms of k. If damping ratio is assumed as 0.05 and $\frac{k}{m}$ is assumed as $200(\text{rad/s})^2$, find the peak and rms displacements of x for El Centro earthquake by (i) Duhamel integral; (ii) frequency domain analysis using FFT; and (iii) state space analysis in time domain.

Figure 3.22

3.14. A rigid pipe embedded within non uniform soil is modeled as lumped mass system as shown in Figure 3.23. The soil stiffness and damping are modeled with springs and dash pots. If the soil damping ratio is assumed to be 0.06 and $\frac{k}{m}$ is assumed as $50(\text{rad/s})^2$, then find the peak and rms responses of y and θ to El Centro earthquake by (i) direct frequency domain analysis using FFT, and (ii) direct time domain analysis using Newmark's-β-method.

Figure 3.23

3.15. Figure 3.24 shows the stick model for a building shear frame. Shear stiffness coefficients are shown in the figure. The base of the frame is subjected to both rotational and translational ground motions. Using the direct frequency domain analysis, find the peak values of moment at the base and dusplacement of the top mass. Take the ground motion as El Centro earthquake and the rotational component of ground motion as translational component divided by 50. Take $\frac{k}{m} = 50(\text{rad/s})^2$ and the damping ratio as 0.02.

Figure 3.24

3.16. Figure 3.25 shows the two-storey shear frame which is subjected to different excitations at the three supports produced due to an arbitrarily assumed time lag in earthquake ground motion. Obtain the peak and rms values of the top displacement for El Centro ground motion by (i) direct time domain analysis using Newmark's-β-method, and (ii) state space analysis in time domain. Assume $\frac{k}{m} = 60\left(\text{rad/s}\right)^2$; time lag between supports as 2.5s; and modal damping ratio as 0.05.

Figure 3.25

3.17. For the frame with a secondary system attached to the top floor as shown in Figure 3.26, find the peak value of the displacement of the mass of the secondary system for different support excitations as shown in the figure by (i) direct frequency domain analysis using FFT, and (ii) state space analysis in frequency domain. Assume earthquake ground motion as El Centro earthquake; time lag as 5s; $\frac{k}{m} = 60\left(\text{rad/s}\right)^2$; damping ratio for the frame and secondary system as 0.02. Compare the result with that obtained for no time lag *i.e.*, the same excitations at the supports.

Figure 3.26

3.18. Using the modal time domain and frequency domain analyses, find the peak values of displacement of the top floor and the first storey drift of the frame shown in Figure 3.27 for El Centro earthquake excitation. Compare between the results obtained by considering contributions of the first two modes only and all modes. Also, find the times at which the peak values of the top storey displacement takes place when only two modes are considered and when all modes are considered. Assume modal damping ratio as 0.05 and $\frac{k}{m} = 40\left(\text{rad/s}\right)^2$.

Figure 3.27

3.19. Using the modal time domain analysis, find the rms and peak values of the dynamic degrees of freedom of a simplified model of a cable stayed bridge shown in Figure 3.28. Use El Centro earthquake with time lag between supports 4 to 5 as 2.5s; 5 to 6 as 5s and 6 to 7 as 2.5s. Assume the following relationships:
$\frac{AE}{l_1} = \frac{12EI}{s^3}$; $m_1 = \frac{1}{3}m$; $\frac{12EI}{s^3 m} = 20\left(\text{rad/s}\right)^2$; modal damping ratio as 0.05; $(EI)_{tower}=0.25(EI)_{deck}=0.25EI$; $(AE/410)_{deck}=0.8(AE/L_1)_{cable}$; s=125; l_1 is the length of the cable joining the top of the tower to the end of the bridge; m_1 is the mass lumped at the top of the tower and m is the mass lumped at the centre of the bridge.

Figure 3.28

3.20. A six-storey shear frame, as shown in Figure 3.29, is subjected to El Centro ground motion. Using the mode acceleration method, find the rms and peak values of the top floor and the first floor displacements shown in the figure. For the analysis, consider contributions of the first three modes only. Compare the results of the mode acceleration method with those of the mode superposition method considering the contributions of all modes. Assume modal damping ratio to be 0.05 and $\frac{k}{m} = 40\left(\text{rad/s}\right)^2$. Obtain the moment in terms of *k*.

Figure 3.29

3.21. A two-storey 3D frame with rigid slab, as shown in Figure 3.30, is subjected to El Centro earthquake. Using the mode acceleration method, find the peak values of the column moments at *A*. Lateral stiffness of the columns are the same in both directions. Total mass lumped at each floor is taken as m and the mass moment of inertia corresponding to the torsional degree of freedom is taken as $\frac{ml^2}{6}$. Assume modal damping ratio as 0.05; $\frac{k}{m} = 80\left(\text{rad/s}\right)^2$; $m = 10000\,\text{kg}$ and $l = 3\,\text{m}$. Consider first three modes for the response calculation and evaluate the accuracy of the results.

Figure 3.30

3.22. A rigid slab is supported by three columns as shown in Figure 3.31. Columns are rigidly connected to the slab and have the same lateral stiffness *k* in both directions. Total mass of the slab is *m* and $\frac{k}{m} = 80\left(\text{rad/s}\right)^2$. Find the time histories of displacements of the top of the columns when the base of the structure is subjected to two-component ground motions as shown in the figure. Take $\ddot{x}_{xg}$ as the El Centro earthquake and assume $\ddot{x}_{yg} = \frac{1}{2}\ddot{x}_{xg}$. Use direct time domain analysis by employing Newmark's- β -method.

Figure 3.31

Take the relevant figures from the slides or from the reference book

Module 3 – Response Analysis for Specified Ground Motion

Exercise Solution:

ERRATA FOR THE TEXT BOOK

pp 100, 3.2.2 subheading: "Absolute motion" should be "Absolute motions"

pp 108, 2nd para, 2nd line: "the other two DOF locked" should be "the other DOF locked"

pp 101, Equation 3.5b: k and c should be $\frac{k}{m}$ and $\frac{c}{m}$ respectively.

pp 105, Example 3.6: A matrix; 4th diaogonal term should be -0.1025ρ

pp 142, Equation 3.115: $\rho_i = \frac{\sum_{r=1}^{n} m_r \phi_{ir}}{M}$ should be $\rho_i = \frac{\sum_{r=1}^{n} \lambda_i m_r \phi_{ir}}{M}$

in which λ_i is defined by Equation 3.114 for single support excitation.

pp 111, Example 3.4: (i) outside coefficient of K_{rr} will be $\frac{EI}{3.6L}$

(ii) Due to computational mistakes, r is calculated wrongly. It should be $r = \begin{bmatrix} 0.5926 & 0.4074 \\ -0.1389 & 0.1389 \end{bmatrix}$; $K = \frac{EI}{L^3}\begin{bmatrix} 16.03 & 10.68 \\ 10.68 & 129.5 \end{bmatrix}$

Example 3.5: K_{41} should be $-\frac{3}{2}\frac{AE}{l_1}\cos^2\theta$; $K_{71} \neq 0$, but $= -\frac{AE}{2l_1}\cos^2\theta$;

$K_{42} \neq 0$, but $= K_{71}$

Because of these changes, r matrix is changed to

$$r = -\begin{bmatrix} -0.781 & -0.003 & 0.002 & -0.218 \\ -0.218 & 0.002 & -0.003 & -0.781 \\ -0.147 & -0.0009 & 0.0009 & 0.147 \end{bmatrix}$$

ERRATA FOR THE TEXT BOOK (Continued)

Example 3.9: Since r matrix is wrong, the response without time delay for d.o.f 5 is not exactly zero. If the correct K and r matrices (given as above for 3.4 (ii)) is used response for d.o.f 5 will be zero. The pattern of responses .shown in Figure 3.14(a),(e),(d) remain the same with little changes in the values of ordinates, if the correct r matrix is used.

Example 3.11: Since r matrix is wrong, the ordinates of the figures (Figures 3.18-.19) change, if the correct r matrix (Example 3.5 as above) is used. The pattern of the responses remain nearly the same. With correct r matrix, peak values of z_1, z_2 and z_3 (given in Table 3.4) change to 0.0377m, 00. 0.05m and 0.02m respectively.

Refer to the exercise problem 3.13 and Figure 3.22.

The kinematic degrees of freedom are θ_B, θ_C and x. $BC = 1.155\rho$, $CD = 3.154\rho$, $OB = 2.31l$ and $OC = 2l$ (θ is the instantaneous centre of rotation). The stiffness matrix corresponding to the degrees of freedom is given below

$$\boldsymbol{K} = k\begin{bmatrix} 0.622l^2 & & l \\ 0.144l^2 & 0.447l^2 & \\ -0.312l & 0.112l & 1.948 \end{bmatrix}\begin{matrix} \theta_B \\ \theta_C \\ x \end{matrix}$$

Condensed stiffness corresponding to the d.o.f x i.e., the equivalent stiffness is 1.709*k*. Equivalent mass corresponding to the d.o.f x is obtained by finding inertia force at x for unit acceleration given at x. The equivalent mass is determined as 2.33m (see the method given in Appendix 3A).

$$\omega_n = \sqrt{\frac{1.709k}{2.33m}} = 12.11 \text{rad s}^{-1}$$

The peak and rms values of the displacement x are shown in Table 3.8 and the time history of displacement x obtained by state space analysis is shown in Fig. 3.38.

Table 3.8 Peak and rms values of x obtained by different methods

Method	Rms (m)	Peak (m)
Duhamel	0.0124	0,0623
FFT	0.012188	0.06151
State space	0.0121360	0.06128

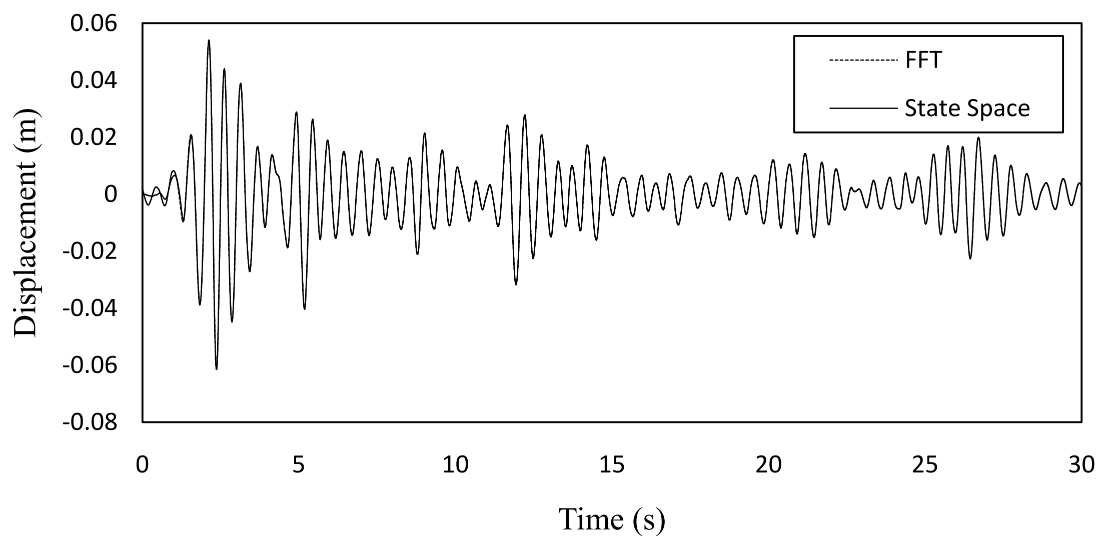


Figure 3.38 Time history of displacement x obtained by state space and frequency domain analysis.

Refer to the exercise problem 3.14 and Figure 3.23.

The stiffness matrix corresponding to the degrees of freedom y and θ is given by

$$\boldsymbol{K} = \begin{bmatrix} 6.7 & \\ -1.65 & 54.375 \end{bmatrix} k$$

$$\boldsymbol{M} = \begin{bmatrix} 3 & \\ & 24.75 \end{bmatrix} m$$

The eigen value analysis provides

$$\lambda_{1,2} = 2.036\frac{k}{m}, 2.329\frac{k}{m}$$

For $\frac{k}{m} = 50$; $\omega_1 = 10.09\,\text{rad s}^{-1}$ $\qquad \omega_2 = 10.936\,\text{rad s}^{-1}$

$$\alpha = \frac{0.06 \times 2 \times (10.09)(10.936)}{10.09 + 10.936} = 0.631$$

$$\beta = \frac{0.06 \times 2}{10.09 + 10.936} = 0.006$$

$$\boldsymbol{C} = 0.631\boldsymbol{M} + 0.006\boldsymbol{K}$$

$$= \begin{bmatrix} 1.893 & \\ & 15.62 \end{bmatrix} m + \begin{bmatrix} 0.04 & -0.01 \\ -0.01 & 0.326 \end{bmatrix} k$$

$$\boldsymbol{C} = \begin{bmatrix} 3.893 & -0.5 \\ -0.5 & 32 \end{bmatrix} m$$

The equation of motion takes the form

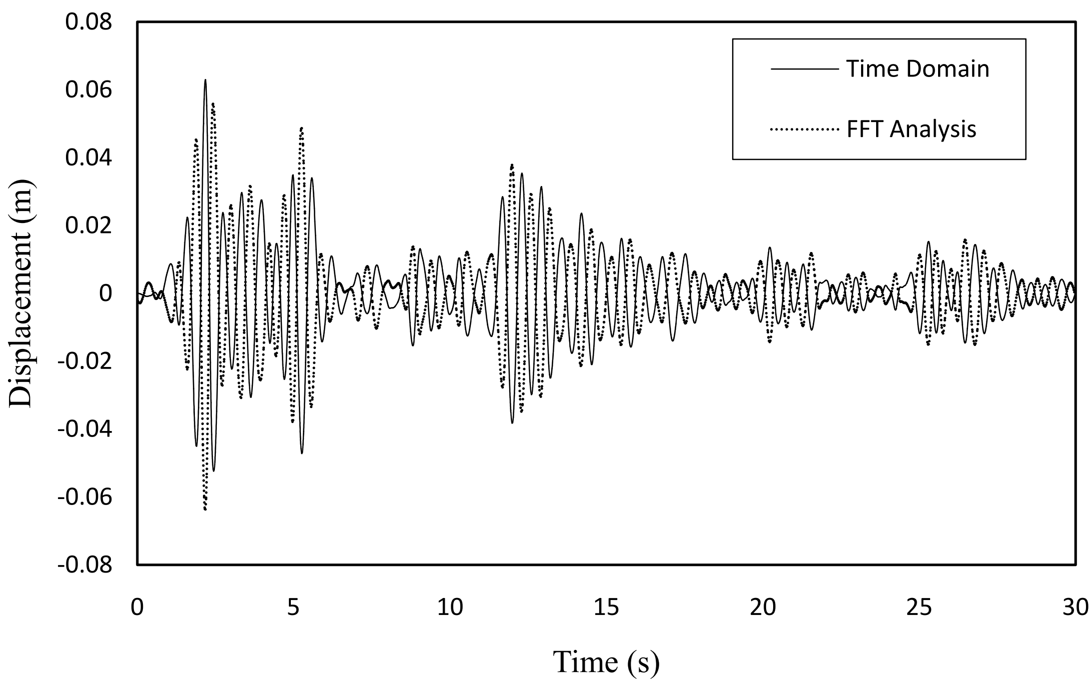
$$\begin{bmatrix} 3 & \\ & 24.75 \end{bmatrix} \ddot{\boldsymbol{x}} + \begin{bmatrix} 3.893 & -0.5 \\ -0.5 & 0.32 \end{bmatrix} \dot{\boldsymbol{x}} + \begin{bmatrix} 335 & -83 \\ -83 & 2719 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 3 & \\ & 24.75 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \ddot{x}_g$$

$$\boldsymbol{x}^T = \begin{bmatrix} y & \theta \end{bmatrix}$$

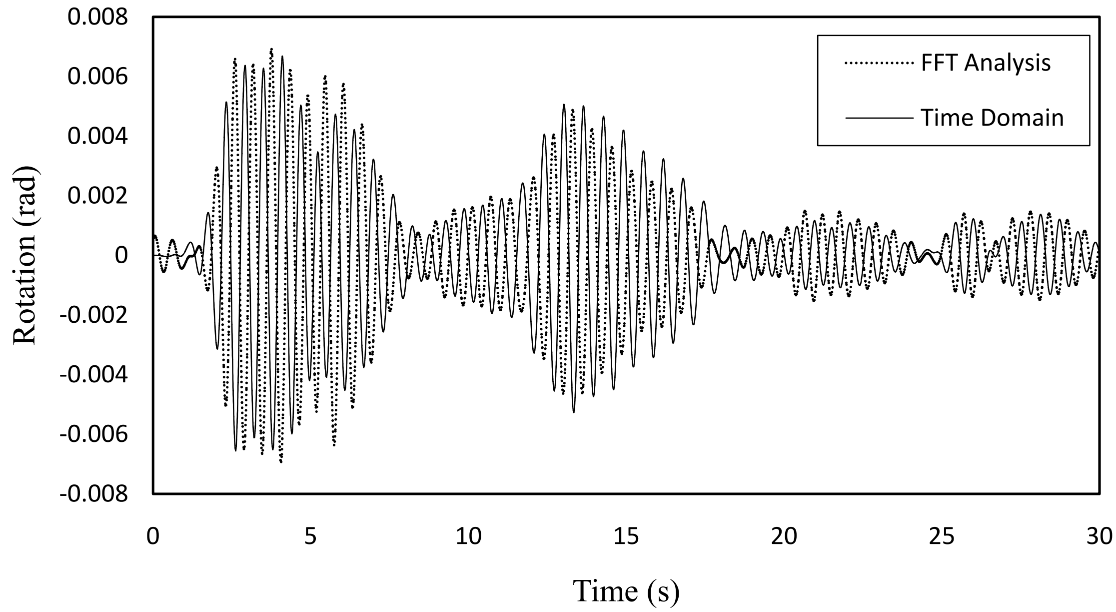
The peak and rms values of the responses are shown in Table 3.9. The time histories of responses obtained by both methods are shown in Figure 3.39

Table 3.9 Peak and rms values of responses

Method	y(m)		θ (rad)	
	peak	**rms**	**peak**	**rms**
Direct Freq.	0.063	0.01275	0.0069	0.0021
Newmark's	0.0629	0.0127	0.0066	0.00206



(a)



(b)

Figure 3.39 Time histories of responses (a) y-displacement (b) rotation-θ

Refer to the exercise problem 3.15 and Figure 3.24

Absolute accelerations of the masses at d.o.f 1 and 2 are

$$\ddot{x}_{a_1} = \ddot{x}_1 + \ddot{x}_g + 6\ddot{\theta}_0$$

$$\ddot{x}_{a_2} = \ddot{x}_2 + \ddot{x}_g + 3\ddot{\theta}_0$$

Equations of motion for the system take the form

$$\boldsymbol{M}\ddot{\boldsymbol{x}}_a + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = 0$$

$$\boldsymbol{M} = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} m \qquad \boldsymbol{C} = \alpha\boldsymbol{M} + \beta\boldsymbol{K} \qquad \boldsymbol{K} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} k$$

Solution of the Eigen value problem provides

$\omega_1 = 5\,\text{rad s}^{-1}$, $\omega_2 = 10\,\text{rad s}^{-1}$

$$\alpha = \frac{0.02 \times 2(5)(10)}{5+10} = \frac{2}{15} = 0.133 \qquad \beta = \frac{0.02 \times 2}{5+10} = 0.00267$$

Equations of motion can be written as

$$\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}\ddot{\boldsymbol{x}} + \begin{bmatrix} 0.2665 & -0.1335 \\ -0.1335 & 0.6665 \end{bmatrix}\dot{\boldsymbol{x}} + \begin{bmatrix} 50 & -50 \\ -50 & 150 \end{bmatrix}\boldsymbol{x} = -\begin{Bmatrix} \ddot{x}_g + 6\ddot{\theta}_0 \\ 2\ddot{x}_g + 3\ddot{\theta}_0 \end{Bmatrix} = -\begin{Bmatrix} 1.12 \\ 2.06 \end{Bmatrix}\ddot{x}_g$$

Frequency response function matrix is given by

$$H(\omega)=\left\{\begin{bmatrix}50-\omega^2 & -50\\ -50 & 150-2\omega^2\end{bmatrix}+i\begin{bmatrix}0.2665 & -0.1335\\ -0.1335 & 0.6665\end{bmatrix}\omega\right\}$$

Using direct frequency domain analysis the responses may be obtained by Equation 3.97 in frequency domain. IFFT of the responses provides the responses in time domain.

Bending moment at the base is

$$\boldsymbol{M}(t)=6kx_2(t)$$

The peak value of $x_1(t)$=-0.222m

The peak value of

$$\boldsymbol{M}(t)=-0.10649\times 2k=-0.10649\times 2k=-0.10649\times 2(50m)=-10.649m$$

Refer to the exercise problem 3.16 and Figure 3.25.

The stiffness matrix corresponding to the d.o.f including the support d.o.f (x_3, x_4, x_5 numbered from left to right)

$$\left[\begin{array}{cc|ccc}5k & -2k & -k & -k & -k\\ -2k & 2k & 0 & 0 & 0\\ \hline -k & 0 & k & 0 & 0\\ -k & 0 & 0 & k & 0\\ -k & 0 & 0 & 0 & k\end{array}\right]\begin{Bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{Bmatrix}$$

$$\boldsymbol{r}=-\boldsymbol{K}_{xx}^{-1}\boldsymbol{K}_{xg}=-\begin{bmatrix}5 & -2\\ -2 & 2\end{bmatrix}^{-1}\begin{bmatrix}-1 & -1 & -1\\ 0 & 0 & 0\end{bmatrix} \qquad \boldsymbol{M}=\begin{bmatrix}2 & \\ & 1\end{bmatrix}m$$

$$=-\begin{bmatrix}-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}\\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}\end{bmatrix}$$

Natural frequencies of the system are obtained as $\omega_1=6.988\,\mathrm{rad\,s^{-1}}$, $\omega_1=14.8\,\mathrm{rad\,s^{-1}}$

The values of α and β are calculated as $\alpha=0.477$, $\beta=0.0046$

Equations of motion take the form

$$\begin{bmatrix} 2 & \\ & 1 \end{bmatrix}\ddot{\boldsymbol{x}} + \begin{bmatrix} 2.334 & -0.552 \\ -0.552 & 1.23 \end{bmatrix}\dot{\boldsymbol{x}} + \begin{bmatrix} 300 & -120 \\ -120 & 120 \end{bmatrix}\boldsymbol{x} = \begin{matrix} -0.67\left(\ddot{x}_{g_1} + \ddot{x}_{g_2} + \ddot{x}_{g_3}\right) \\ -0.33\left(\ddot{x}_{g_1} + \ddot{x}_{g_2} + \ddot{x}_{g_3}\right) \end{matrix}$$

State space equation is

$\dot{Y} = AY + F$

$$A = \begin{bmatrix} 0 & I \\ -KM^{-1} & -CM^{-1} \end{bmatrix} \qquad F = \begin{Bmatrix} 0 \\ 0 \\ -0.67\left(\ddot{x}_{g_1} + \ddot{x}_{g_2} + \ddot{x}_{g_3}\right) \\ -0.33\left(\ddot{x}_{g_1} + \ddot{x}_{g_2} + \ddot{x}_{g_3}\right) \end{Bmatrix}$$

Because of the time lag of 2.5s between the supports, the total time history of excitation at each support is taken as 35s and is constructed as explained in Example 3.8.

The time histories of the top displacement obtained by direct time domain (T) and state space time domain (S) analyses are given below.

The peak value of top displacement = 0.0682m (T); 0.0689 m(S)

The rms value of top displacement = 0.0211m (T); 0.02004 m(S)

Refer to the exercise problem 3.17 and Figure 3.26.

The stiffness matrix for non support degrees of freedom $\boldsymbol{K}_{xx}$ and the coupling matrix $\boldsymbol{K}_{xg}$ between support and non support degrees of freedom are

$$\boldsymbol{K}_{xx} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2.187 & -0.187 \\ 0 & -0.187 & 0.187 \end{bmatrix}k \qquad \boldsymbol{K}_{xg} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}k$$

$$\boldsymbol{r} = -\boldsymbol{K}_{xx}^{-1}\boldsymbol{K}_{xg} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \qquad \boldsymbol{M} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.25 \end{bmatrix}m$$

Frequencies of the system are calculated as $\omega_1 = 5.44\,\mathrm{rad\,s^{-1}}$, $\omega_2 = 8.28\,\mathrm{rad\,s^{-1}}$, $\omega_3 = 17.83\,\mathrm{rad\,s^{-1}}$

Taking the first two frequencies

$\alpha = 0.1314$; $\beta = 0.0029$

The equations of motion for the system take the form

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.25 \end{bmatrix}\ddot{\boldsymbol{x}} + \begin{bmatrix} 0.831 & -0.35 & 0 \\ -0.35 & 0.514 & -0.033 \\ 0 & -0.033 & 0.065 \end{bmatrix}\dot{\boldsymbol{x}} + \begin{bmatrix} 240 & -120 & 0 \\ -120 & 131.22 & -1.122 \\ 0 & -1.122 & 1.122 \end{bmatrix}\boldsymbol{x} = -\begin{bmatrix} 0.5\left(\ddot{x}_{g_1} + \ddot{x}_{g_2}\right) \\ 0.5\left(\ddot{x}_{g_1} + \ddot{x}_{g_2}\right) \\ 0.5\left(\ddot{x}_{g_1} + \ddot{x}_{g_2}\right) \end{bmatrix}$$

$\ddot{x}_{g1}$ and $\ddot{x}_{g2}$ have a time lag of 5 s.

The time histories of the displacement of the top mass obtained by direct frequency domain (D) and state space frequency domain (S) are shown in Figure 3.40.

The peak value (time lag) = 0.2878m (D); 0.2191m (S)

The peak value (no time lag) = 0.26801m (D); 0.2824m (S)

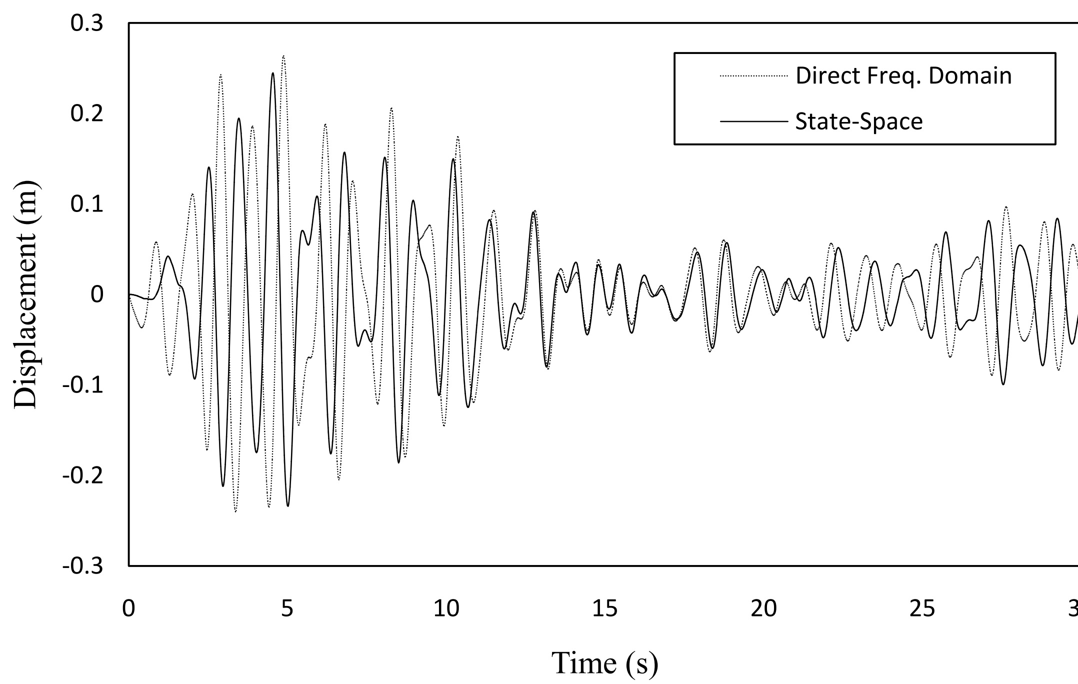


Figure 3.40 Time history of the displacement of the secondary system

Refer to the exercise problem 3.18 and Figure 3.27.

Stiffness matrix for the system is

$$\boldsymbol{K} = \begin{bmatrix} 7 & -3 & 0 & 0 \\ -3 & 5 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} k \qquad \boldsymbol{M} = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{bmatrix} m$$

The natural frequencies of the system are

$\omega_1 = 2.54\,\text{rad s}^{-1}$, $\omega_2 = 3.91\,\text{rad s}^{-1}$, $\omega_3 = 4.52\,\text{rad s}^{-1}$, $\omega_4 = 13.71\,\text{rad s}^{-1}$

$\phi_1^T = [-0.183 \quad -0.407 \quad -0.677 \quad -1]$; $\qquad \phi_2^T = [-0.552 \quad -0.967 \quad -0.745 \quad 1]$

$\phi_3^T = [-0.7635 \quad -0.627 \quad 1 \quad -0.283]$; $\qquad \phi_4^T = [-1 \quad 0.789 \quad -0.254 \quad 0.0308]$

$\lambda_1 = -1.367$, $\lambda_2 = -0.4525$, $\lambda_3 = -0.3274$, $\lambda_4 = -0.25$

Time histories of top floor displacement obtained by considering the first two modes and all modes are:

Top story peak displacement = 0.3627m (2modes); 0.3587m (all modes)

Occurrence peaks of displacement = 6.8sec (2modes); 6.835s (all modes)

First story peak displacement = 0.1565 m (2modes); 0.1525 (all modes)

Refer to the exercise problem 3.19 and Figure 3.28.

The length of the cable are computed as $l_1 = 94.34m$ $\cos\theta_1 = 0.848$ $sin\theta_1 = 0.53$ l_2 (length of the cable joining the tower top to the centre of the bridge) = 134.63 m

$cos\theta_2 = 0.9286$ $sin\theta_2 = 0.371$; tower bases are fixed (Figure 3.28 does not show them properly).

$$K_{11} = \frac{3 \times 0.25EI}{(80)^3} + \frac{AE}{l_1} cos^2\theta_1 + \frac{AE}{l_2} cos^2\theta_2$$

$$= \frac{0.75EI}{(80)^3} + \frac{12EI}{(125)^3}(0.6098) + \frac{12EI}{(125)^3}(0.7)(0.862)\,[0.238 + 0.6098 + 0.6034]\frac{12EI}{(125)^3} = 580m$$

$K_{21} = 0$;

$$K_{31} = -\frac{AE}{l_2} cos\theta_2 sin\theta_2 = -\frac{AE}{l_1}\frac{l_1}{l_2} cos\theta_2 sin\theta_2 = -\frac{AE}{l_1}\frac{94.34}{134.63}(0.9286)(0.371) = -0.241\frac{AE}{l_1} = -96.4m$$

$$K_{41} = -\frac{AE}{l_1} cos^2\theta_1 - \frac{1}{2}\frac{AE}{l_2} cos^2\theta_2 = -\frac{AE}{l_1}(0.848)^2 - \frac{AE}{l_2}(0.9286)^2 = -409m$$

$$K_{51} = -\frac{3\left(EI\right)_{tower}}{\left(80\right)^3} = -\frac{0.75EI}{\left(80\right)^3} = \frac{0.75EI \times 12}{12 \times \left(125\right)^3} \times \frac{\left(125\right)^3}{\left(80\right)^3} = \frac{0.75}{12} \times \frac{\left(125\right)^3}{\left(80\right)^3} \times 400m = -95.4m$$

$$K_{61} = K_{61} = K_{91} = K_{101} = 0\,; \quad K_{71} = -\frac{1}{2}\frac{AE}{l_2}\cos^2\theta_2 = -120.8m$$

$$K_{22} = K_{11} \qquad K_{32} = -K_{31} \qquad K_{42} = -120.8m \qquad K_{52} = 0$$

$$K_{62} = K_{51} \qquad K_{72} = K_{41} \qquad K_{82} = K_{92} = K_{102} = 0$$

$$K_{33} = \frac{24EI}{\left(125\right)^3} + \frac{2AE}{l_2} sin^2\theta_2 = \frac{24EI}{\left(125\right)^3} + \frac{2AE}{94.34}\left(\frac{94.34}{134.63}\right)\left(0.371\right)^2 = 800m\left(1+0.096\right) = 876.8m$$

$$K_{43} = K_{53} = K_{63} = K_{73} = 0$$

$$K_{83} = \frac{-6EI}{\left(125\right)^3} = -25000m \qquad K_{93} = 0 \qquad K_{103} = 25000m$$

$$K_{44} = \left(\frac{AE}{400}\right)_{deck} + \frac{2AE}{L_2}\cos^2\theta_2 + \frac{AE}{L}\cos^2\theta_1 = 0.8\left(\frac{AE}{l_1}\right)_{cable} + 2\times 241.7m + 288m = 1091m$$

$$K_{74} = -1091m \qquad K_{54} = K_{64} = K_{84} = K_{94} = K_{104} = 0$$

$$K_{55} = -K_{51} = 95.4m \qquad K_{65} = K_{75} = K_{85} = K_{95} = K_{105} = 0$$

$$K_{66} = 95.4m \qquad K_{76} = K_{86} = K_{96} = K_{106} = 0$$

$$K_{77} = K_{44} = 1091m \qquad K_{87} = K_{97} = K_{107} = 0$$

$$K_{88} = \frac{4EI}{125} + \frac{3EI}{80} = \frac{4EI}{125} + \frac{3EI}{125} \times \frac{125}{80} = \frac{8.68}{125} EI = \frac{8.68}{\left(125\right)^3} \times \frac{\left(125\right)^2}{12} \times 12EI = 452 \times 10^4 m$$

$$K_{98} = \frac{2EI}{125} = \frac{2EI}{125} = 104 \times 10^4 m$$

$$K_{108} = 0$$

$$K_{99} = \frac{8EI}{125} = 416 \times 10^4 m$$

$$K_{109} = \frac{2EI}{125} = 104 \times 10^4 m$$

$$K_{1010} = K_{88}$$

$$K_{\theta\theta}=\begin{bmatrix}452 & 104 & 0\\ 104 & 416 & 104\\ 0 & 104 & 452\end{bmatrix}\times 10^4 m$$

$$K_{\Delta\Delta}=\begin{bmatrix}0.58 & & & & & & \\ 0 & 0.58 & & & & & \\ -0.096 & 0.096 & 0.876 & & & & \\ -0.409 & -0.121 & 0 & 1.09 & & & \\ -0.095 & 0 & 0 & 0 & 0.095 & & \\ 0 & -0.095 & 0 & 0 & 0 & 0.095 & \\ -0.121 & -0.409 & 0 & -1.09 & 0 & 0 & 1.09\end{bmatrix}\times 10^3 m$$

$$K_{\theta\Delta}=\begin{bmatrix}0 & 0 & -2.5 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -2.5 & 0 & 0 & 0 & 0\end{bmatrix}\times 10^4 m$$

Condensing out the rotational stiffness to find $\bar{K}_{\Delta\Delta}$ and then partitioning it to identify the sub matrices for support and non support degrees of freedom, the matrix r is obtained as

$$r=-K_{ss}^{-1}K_{sn}=\begin{bmatrix}0.719 & 0.1684 & -0.0046 & 0.1947\\ 0.1947 & -0.0046 & 0.1684 & 0.719\\ 0.0839 & 0.0277 & -0.0277 & -0.0839\end{bmatrix}$$

where s and n refers to support and non support d.o.f

$$\boldsymbol{K}_{ss}=\begin{bmatrix}580 & & Sym\\ 0 & 580 & \\ -96 & 96 & 600\end{bmatrix}m \qquad \boldsymbol{M}=\begin{bmatrix}20 & & \\ & 20 & \\ & & 60\end{bmatrix}m$$

The mode shapes and frequencies are

$$\phi_1^T=\begin{bmatrix}0.2426 & -0.2426 & 1\end{bmatrix},\quad \omega_1=3.055\,\text{rad s}^{-1}$$

$$\phi_2^T=\begin{bmatrix}-1 & -1 & 0\end{bmatrix},\quad \omega_2=5.384\,\text{rad s}^{-1}$$

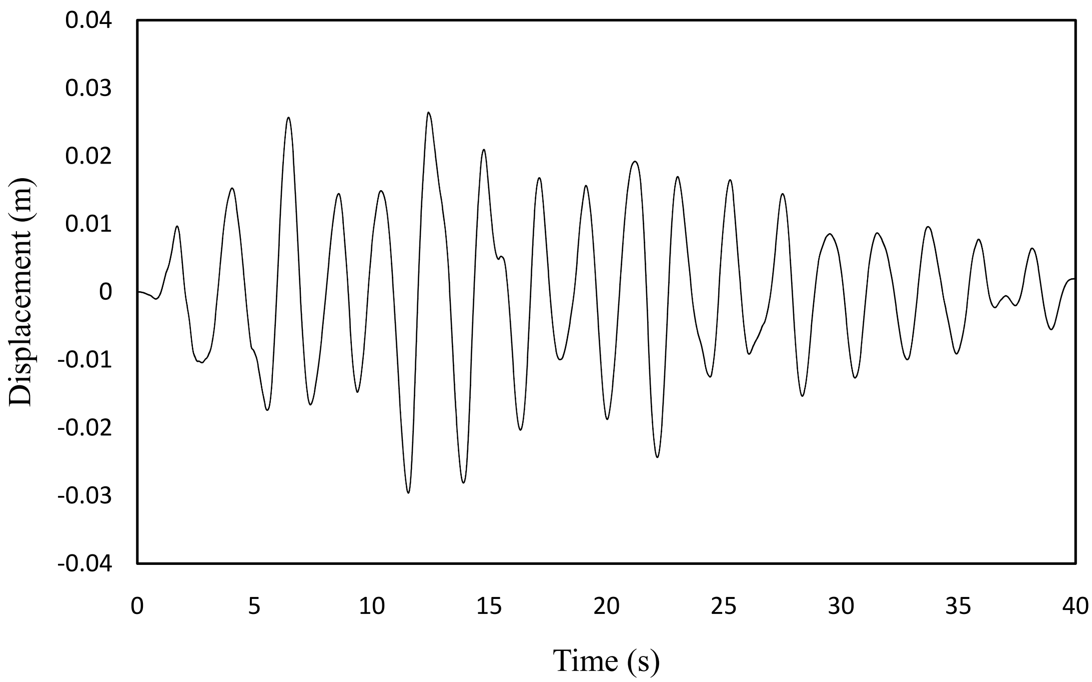
$$\phi_3^T=\begin{bmatrix}1 & -1 & -0.1617\end{bmatrix},\quad \omega_3=5.457\,\text{rad s}^{-1}$$

$$\lambda=\begin{bmatrix}0.1216 & 0.0401 & -0.0401 & -0.1216\\ -0.4569 & -0.082 & -0.082 & -0.4569\\ 0.2326 & 0.0767 & -0.0767 & -0.2326\end{bmatrix};\qquad \lambda_{ik}\ \begin{matrix}i=1\cdots\cdots 3\\ k=1\cdots\cdots 4\end{matrix}$$

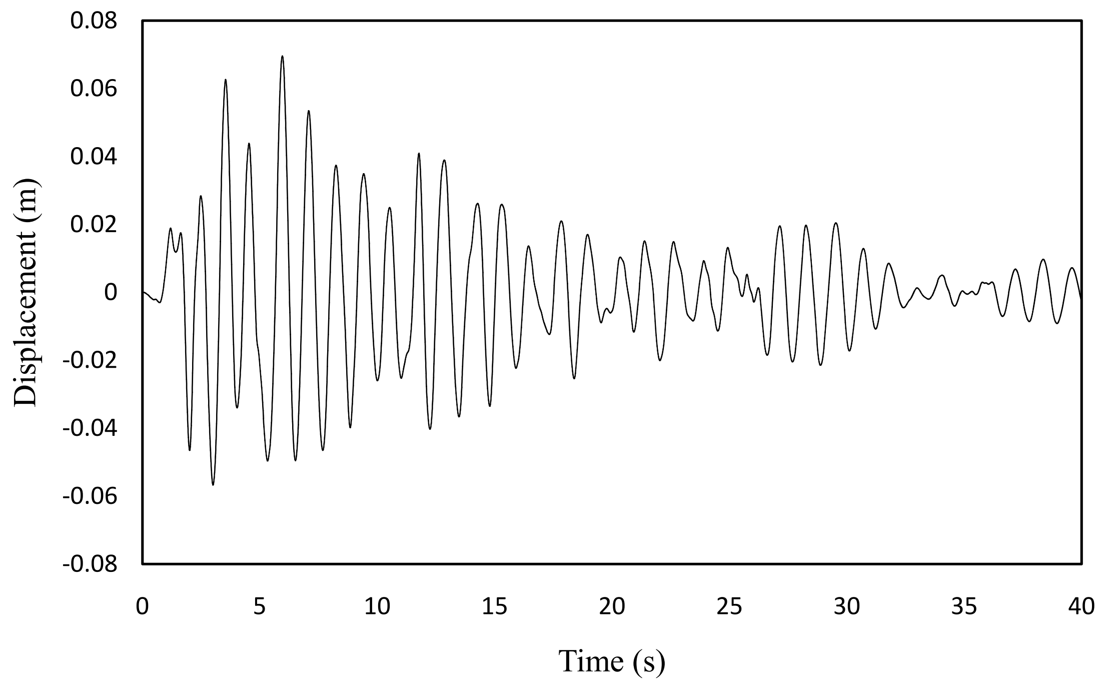
The time histories of excitations at the supports are of 45s duration and are constructed as explained in Example 3.8.
The time histories of the vertical displacement of the deck and the top of the left tower are shown in Figure3.41.
The rms and peak values of the three degrees of freedom are given as below:

Table 3.10: Rms and peak values of responses

d.o.f	**rms (m)**	**peak (m)**
1	0.0189	0.0696
2	0.0195	0.0584
3	0.0109	0.0296



(a)



(b)

Figure 3.41 The time histories of displacements (a) top of the left tower (b) centre of the deck

Refer to the exercise problem 3.20 and Figure 3.29.

Stiffness and mass matrices for the frame are:

$$\boldsymbol{K} = \begin{bmatrix} 2 & & & & & \\ -1 & 2 & & & Sym & \\ 0 & -1 & 2 & & & \\ 0 & 0 & -1 & 2 & & \\ 0 & 0 & 0 & -1 & 2 & \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} k \qquad M = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} m$$

The natural frequencies of the frame are:

$\omega_1 = 1.525\,\text{rad}\,\text{s}^{-1}$, $\omega_2 = 4.49\,\text{rad}\,\text{s}^{-1}$, $\omega_3 = 7.185\,\text{rad}\,\text{s}^{-1}$, $\omega_4 = 9.47\,\text{rad}\,\text{s}^{-1}$, $\omega_5 = 11.2\,\text{rad}\,\text{s}^{-1}$,

$\omega_6 = 12.3\,\text{rad}\,\text{s}^{-1}$

$$\phi_1^T = \begin{bmatrix} -0.241 & -0.468 & -0.668 & -0.829 & -0.942 & -1 \end{bmatrix}$$

$$\phi_2^T = \begin{bmatrix} 0.668 & 1 & 0.829 & 0.241 & -0.468 & -0.942 \end{bmatrix}$$

$$\phi_3^T = \begin{bmatrix} -0.942 & -0.668 & 0.468 & 1 & 0.241 & -0.829 \end{bmatrix}$$

$$\phi_4^T = \begin{bmatrix} 1 & -0.241 & -0.941 & 0.468 & 0.829 & -0.668 \end{bmatrix}$$

$\phi_5^T = \begin{bmatrix} 0.829 & -0.942 & 0.242 & 0.668 & -1 & 0.468 \end{bmatrix}$

$\phi_6^T = \begin{bmatrix} -0.468 & 0.829 & -1 & 0.942 & -0.668 & 0.241 \end{bmatrix}$

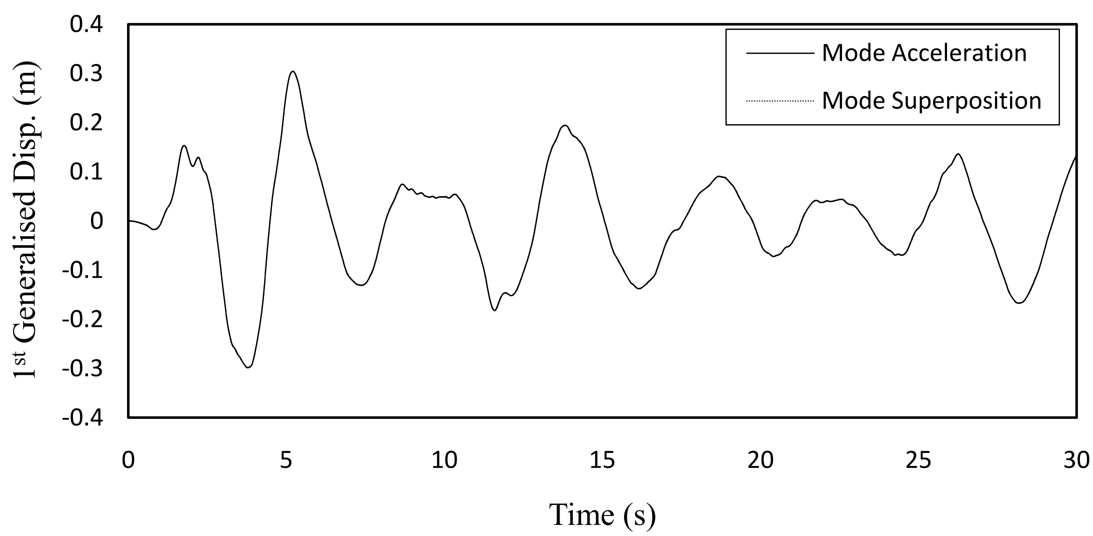
$\lambda_1 = -1.258$, $\lambda_2 = 0.4027$, $\lambda_3 = -0.2212$, $\lambda_4 = 0.1353$, $\lambda_5 = 0.0802$, $\lambda_6 = -0.0376$

Using Equations 3.123 and 3.124 (with the steps given to find out $\bar{R}(t)$, the responses are obtained using mode acceleration approach considering the first three modes. For the mode superposition method, contributions of the first four modes are considered. The results are given below

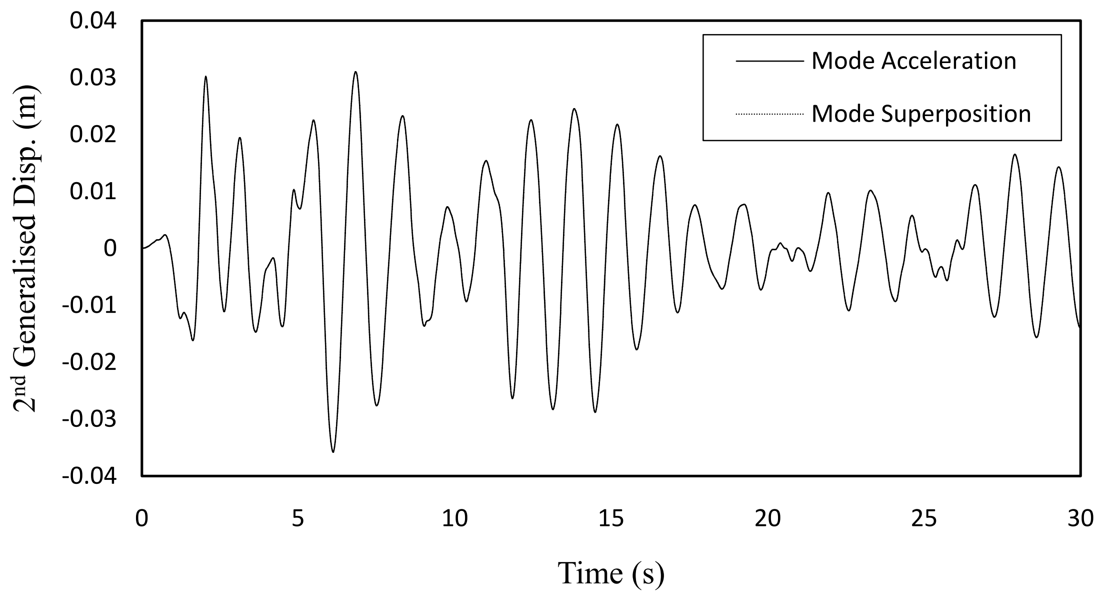
Table 3.11: Rms and peak responses

Floor	**rms (m)**		**peak (m)**	
	Mode accl.	**Mode sup.**	**Mode accl.**	**Mode sup.**
Top	0.10946	0.1095	0.323	0.3245
1st	0.0273	0.0273	0.083	0.085

The time histories of the first two generalized displacements are shown in Figure 3.42. The displacements of the top and the first floors obtained by both mode acceleration and mode superposition method are shown in Figure 3.43.

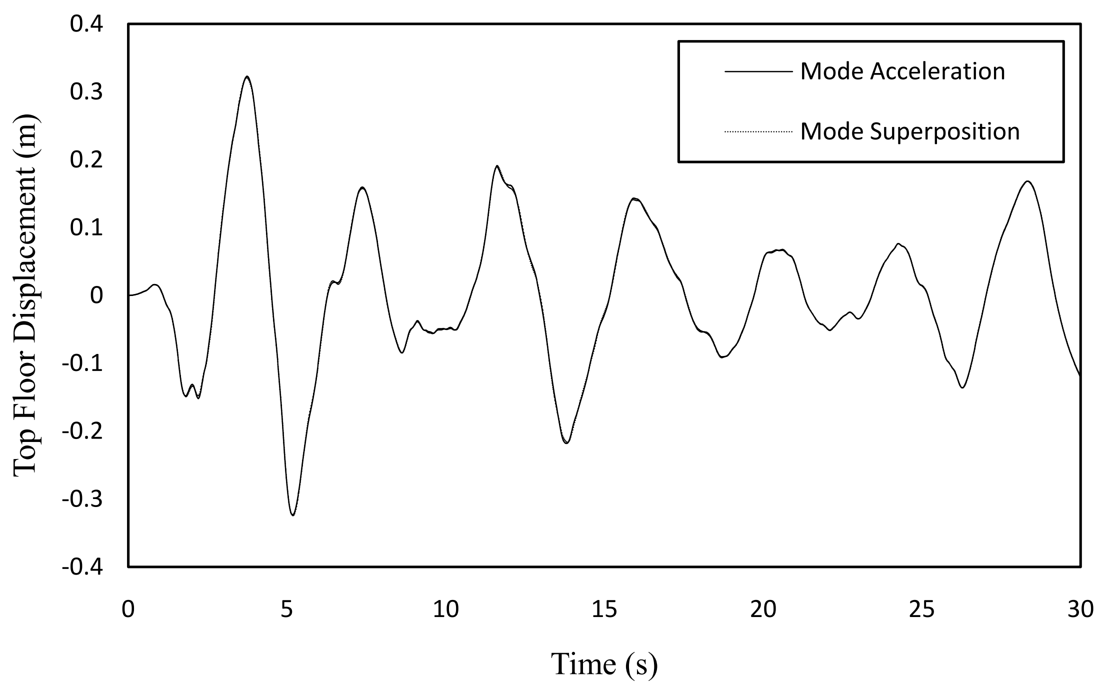


(a)

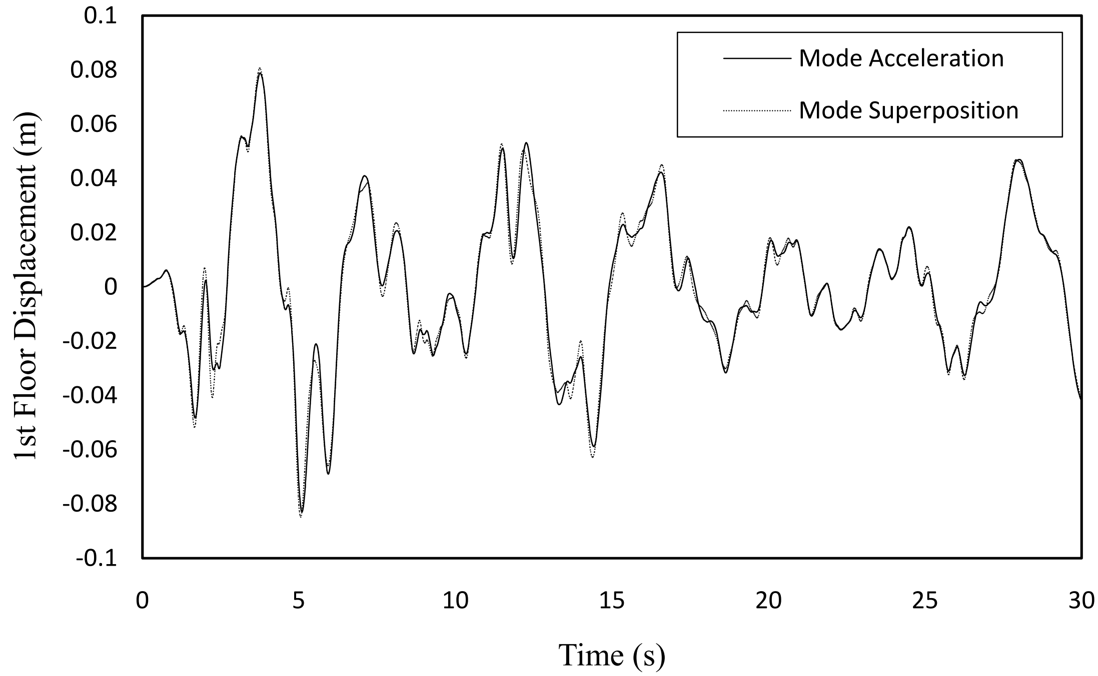


(b)

Figure 3.42 Time histories of generalized displacements (a) first generalized displacement (b) second generalized displacement



(a)



(b)

Figure 3.43 Time histories of displacements (a) top floor (b) first floor

Refer to the exercise problem 3.21 and Figure 3.30.

The stiffness and mass matrices are of the frame are:

$$\boldsymbol{K} = \begin{bmatrix} 15 & & & & & \\ 0 & 15 & & & & \\ 1.5 & 7.5 & 67.5 & & & \\ -7.5 & 0 & -0.75 & 7.5 & & \\ 0 & -7.5 & -3.75 & 0 & 7.5 & \\ -0.75 & -3.75 & -33.75 & 0.75 & 3.75 & 33.75 \end{bmatrix} k$$

$$\boldsymbol{M} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1.5 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1.5 \end{bmatrix} m$$

Frequencies of the system are

$\omega_1 = 14.5\,\text{rad s}^{-1}$, $\omega_2 = 15.14\,\text{rad s}^{-1}$, $\omega_3 = 26.58\,\text{rad s}^{-1}$, $\omega_4 = 38\,\text{rad s}^{-1}$, $\omega_5 = 39.6\,\text{rad s}^{-1}$,

$\omega_6 = 69.6\,\text{rad s}^{-1}$

$$\phi_1^T = \begin{bmatrix} 0.123 & 0.618 & -0.102 & 0.2 & 1 & -0.1664 \end{bmatrix}$$

$$\phi_2^T = \begin{bmatrix} -0.618 & 0.123 & 0 & -1 & 0.2 & 0 \end{bmatrix}$$

$$\phi_3^T = \begin{bmatrix} -0.0296 & -0.148 & -0.618 & -0.048 & -0.24 & -1 \end{bmatrix}$$

$$\phi_4^T = \begin{bmatrix} 0.2 & 1 & -0.166 & -0.124 & -0.618 & 0.103 \end{bmatrix}$$

$$\phi_5^T = \begin{bmatrix} 1 & -0.2 & 0 & -0.618 & 0.124 & 0 \end{bmatrix}$$

$$\phi_6^T = \begin{bmatrix} -0.048 & -0.24 & -1 & 0.0296 & 0.148 & 0.618 \end{bmatrix}$$

$\lambda_1 = 0.2165$, $\lambda_2 = -1.126$, $\lambda_3 = -0.036$, $\lambda_4 = 0.0511$, $\lambda_5 = 0.2657$, $\lambda_6 = -0.0085$

The results of the responses by considering first three modes are:

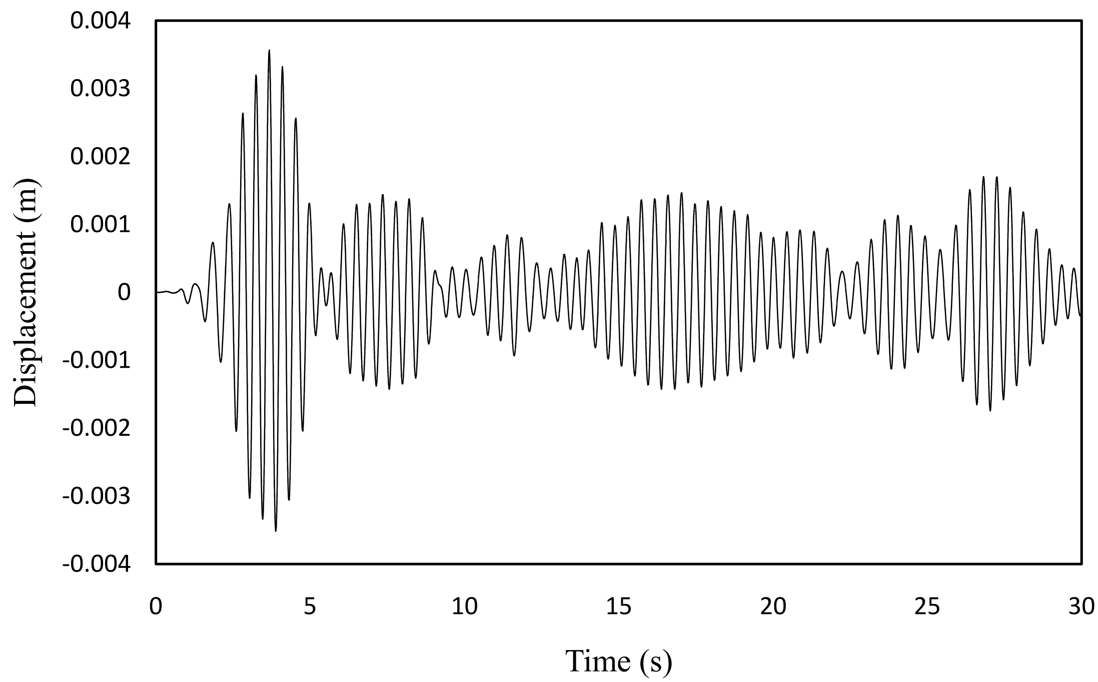
Peak values :

Top floor y-displacement = 0.00356m

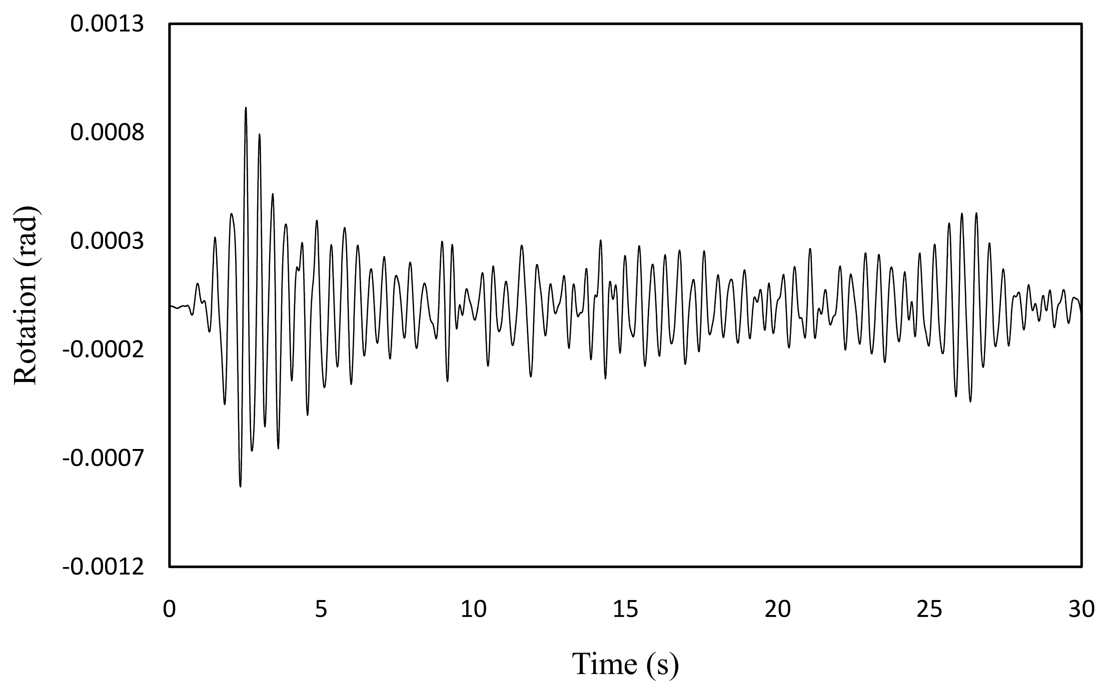
First floor x-displacement = 0.02434m

First floor rotation = 0.0009143 rad

The y-displacement of the top floor and the rotation of the first floor are shown in Figure 3.44



(a)



(b)

Figure 3.44 Time histories of (a) y-displacement of top floor (b) rotation of the first floor

Refer to the exercise problem 3.22 and Figure 3.31.

Due to unit displacement given at the d.o.f one at a time, stiffness and the mass matrices are generated as explained in Appendix 3A.

The stiffness and mass matrices corresponding to the d.o.f are

$$\boldsymbol{K} = \begin{bmatrix} 10 & -7 & 3 \\ -7 & 7 & 3 \\ -3 & 3 & 3 \end{bmatrix} k \qquad \boldsymbol{M} = \begin{bmatrix} \frac{8}{8} & -\frac{5}{3} & -1 \\ -\frac{5}{3} & \frac{8}{3} & 1 \\ -1 & 1 & 1 \end{bmatrix} m \qquad \boldsymbol{I} = \begin{bmatrix} 1 & 1 & 0.5 \end{bmatrix}$$

$\alpha = 0.595$, $\beta = 0.00397$

$$\boldsymbol{C} = \begin{bmatrix} 4.7672 & -3.218 & -1.5492 \\ -3.218 & 3.8132 & 1.5492 \\ -1.5492 & 1.5492 & 1.5492 \end{bmatrix}$$

Frequencies of the system are

$\omega_1 = 9.66\,\mathrm{rad\,s^{-1}}$, $\omega_2 = 15.5\,\mathrm{rad\,s^{-1}}$, $\omega_3 = 18.77\,\mathrm{rad\,s^{-1}}$

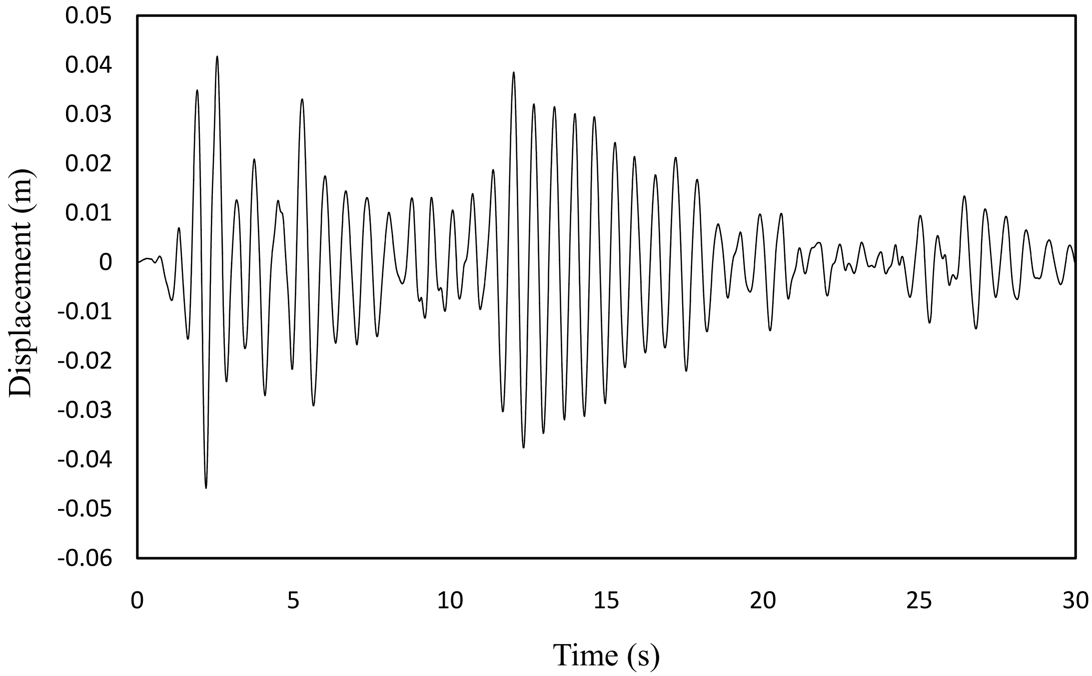
Time histories of displacements of the top of the columns are shown in Figure 3.45.

Peak values of displacements are

For d.o.f 1 = 0.046m

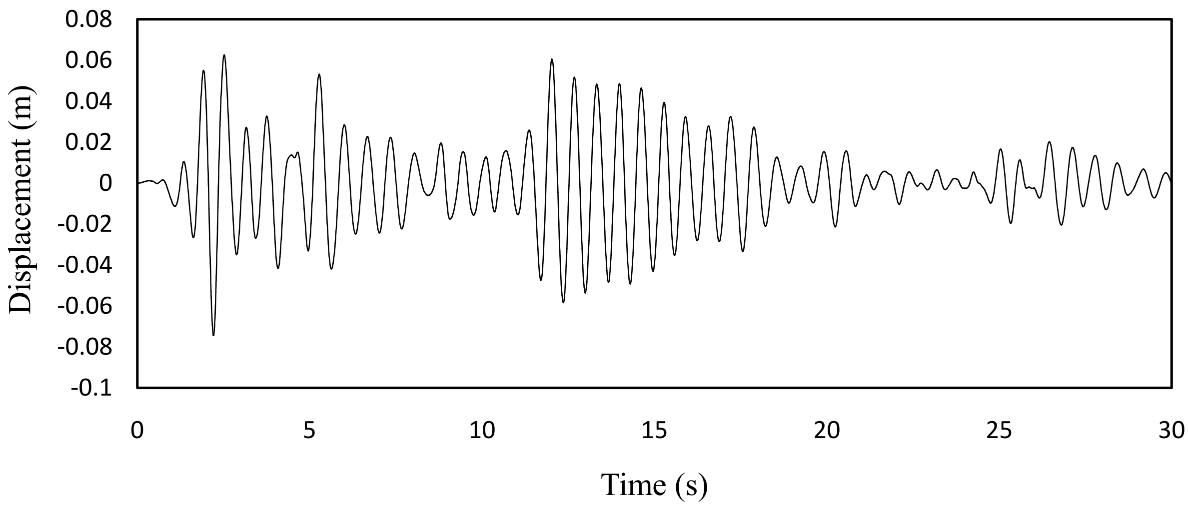
For d.o.f 2 = 0.0744m

For d.o.f 3 = 0.403m



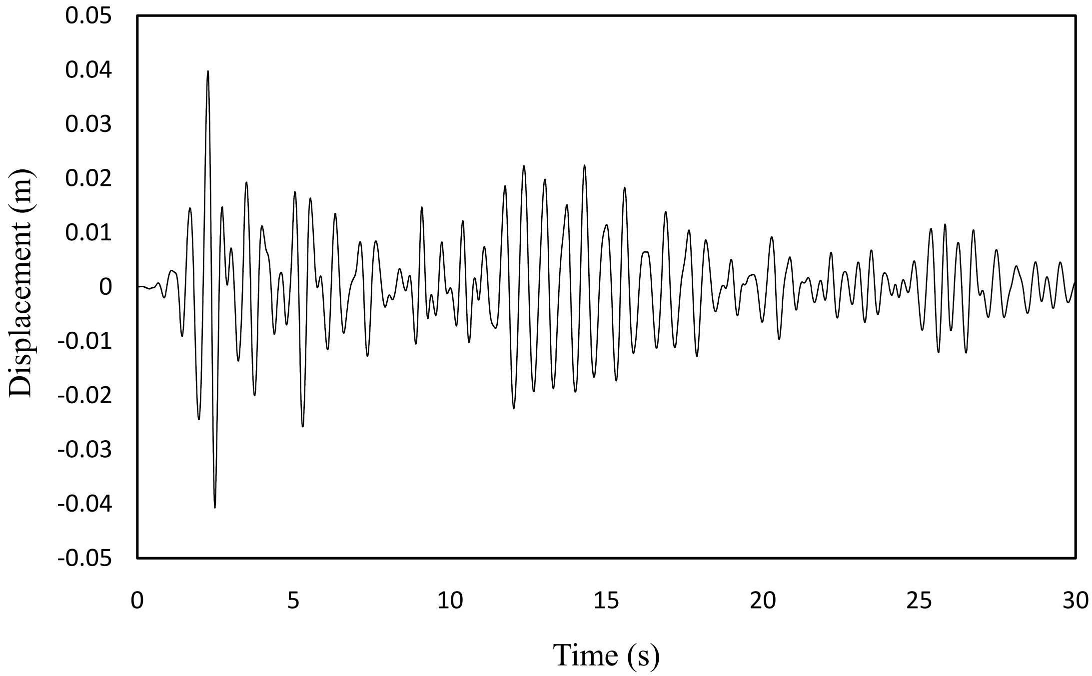
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
Displacement (m)
0
5
10
15
20
25
30
Time (s)

(a)



0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
Displacement (m)
0
5
10
15
20
25
30
Time (s)

(b)



(c)

Figure 3.45 Time histories of (a) d.o.f-1 (b) d.o.f-2 (c) d.o.f-3