Advanced Numerical Analysis for Chemical Engineering
Mid-Term Examination -1 (2 hrs.)

1. Let X represent set of continuous functions on interval $0 \leq t \leq 1$ with inner product defined as

$$\langle \mathbf{x}(t), \mathbf{y}(t) \rangle = \int_0^1 w(t)\mathbf{x}(t)\mathbf{y}(t)dt$$

Given a set of two linearly independent vectors

$$\mathbf{x}^{(1)}(t) = 1; \quad \mathbf{x}^{(2)}(t) = t$$

find orthonormal set of vectors $\mathbf{e}^{(1)}(t)$ and $\mathbf{e}^{(2)}(t)$ if $w(t) = t(1-t)$. (6 marks)

2. Consider system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 3 & 1 & -2 \\ 3 & 1 & -2 \end{bmatrix}$$

   1. What is the dimension of column space of matrix A, i.e. $R(A)$? Find a basis for $R(A)$.
   2. What is the dimension of null space of matrix A, i.e. $N(A)$? Find a basis for $R(A)$. (6 marks)

      Hint: Note that, for a $n \times n$ matrix A

$$\begin{aligned} \dim[R(A)] &= \dim[R(A^T)] \\ \dim[R(A^T)] + \dim[N(A)] &= n \end{aligned}$$

   3. It is proposed to define an inner product as follows

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y}$$

      Is this a valid definition of inner product? Justify your answer. (4 marks)

3. Application of finite difference method to solve a PDE resulted in a set of linear algebraic equation, $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \quad ; \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

It is desired to solve the above equation using Jacobi and Gauss Seidel methods starting from the following initial guess solution

$$\mathbf{x}^{(0)} = \begin{bmatrix} -5 & 2 & -3 & 1 & -3 & 5 \end{bmatrix}^T$$

Will the iterations converge in each case? Justify your answer. (4 marks)

4. Consider the following difference equation initial value problem

$$\mathbf{e}^{(k+1)} = A\mathbf{e}^{(k)} \;\; ; \; A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \quad \mathbf{e}^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   1. Find eigen-values and eigen vectors of A. (3 marks)
   2. Comment upon the asymptotic behavior of the solution $\mathbf{e}^{(k)}$ for large k (i.e. $k \to \infty$). (3 marks)