Advanced Process Control
Computing Examination II (2 hours)
Question Paper A

Consider a system governed by the following set of nonlinear ODEs

$$\frac{d\mathbf{X}_1}{dt} = 1.1226 \times 10^{-2} \left[\sqrt{\mathbf{X}_3} - \sqrt{\mathbf{X}_1}\right] + 8.325 \times 10^{-2}\mathbf{U}_1 \quad (1)$$

$$\frac{d\mathbf{X}_2}{dt} = 7.8859 \times 10^{-3} \left[\sqrt{\mathbf{X}_4} - \sqrt{\mathbf{X}_2}\right] + 6.2812 \times 10^{-2}\mathbf{U}_2 \quad (2)$$

$$\frac{d\mathbf{X}_3}{dt} = -1.1226 \times 10^{-2}\sqrt{\mathbf{X}_3} + 4.7857 \times 10^{-2}\mathbf{U}_2 \quad (3)$$

$$\frac{d\mathbf{X}_4}{dt} = -7.8859 \times 10^{-3}\sqrt{\mathbf{X}_4} + 3.1219 \times 10^{-2}\mathbf{U}_1 \quad (4)$$

Initial state of the plant is

$$\mathbf{X}(0) = \begin{bmatrix} 11 & 13.5 & 2.4 & 3 \end{bmatrix}^T$$

A discrete linear perturbation model for this system in the neighborhood of steady state operating point

$$\overline{\mathbf{X}} = \begin{bmatrix} 12.4 & 12.7 & 1.8 & 1.4 \end{bmatrix}^T \quad \text{and} \quad \overline{\mathbf{U}} = \begin{bmatrix} 3 & 3 \end{bmatrix}^T$$

is given as follows

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.9225 & 0 & 0.1874 & 0 \\ 0 & 0.946 & 0 & 0.1492 \\ 0 & 0 & 0.8046 & 0 \\ 0 & 0 & 0 & 0.8465 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.4003 & 0.0235 \\ 0.0121 & 0.304 \\ 0 & 0.214 \\ 0.1439 & 0 \end{bmatrix} \mathbf{u}(k)$$

Measured outputs are related to the states as follows

$$\mathbf{y}(t) = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \mathbf{v}(k)$$

where $\mathbf{v}(k)$ are zero mean Gaussian white noise sequence with

$$Cov\left[\mathbf{v}(k)\right] = \begin{bmatrix} (0.05)^2 & 0 \\ 0 & (0.06)^2 \end{bmatrix}$$

- **Plant Simulation**: Simulate the plant in open loop for k = 1,2,...100 samples under the following input conditions

– Generate the *known component* of the manipulated input signal, $\mathbf{u}(k)$, as follows

$$\begin{array}{ll} 0 \leq k \leq 30 & \mathbf{u}(k) = \begin{bmatrix} 3 & 3 \end{bmatrix}^T \\ 31 \leq k \leq 65 & \mathbf{u}(k) = \begin{bmatrix} 4 & 2 \end{bmatrix}^T \\ 66 \leq k \leq 100 & \mathbf{u}(k) = \begin{bmatrix} 2 & 4 \end{bmatrix}^T \end{array}$$

– Unmeasured disturbance: The manipulated input entering the plant dynamics is $[\mathbf{u}(k) + \mathbf{w}(k)]$ where $\mathbf{w}(k)$ is a zero mean Gaussian white noise sequences with

$$\mathbf{Q} = Cov\left[\mathbf{w}(k)\right] = \begin{bmatrix} (0.06)^2 & 0 \\ 0 & (0.07)^2 \end{bmatrix}$$

- While simulating the plant dynamics, give user choice to simulate the plant either as

$$\begin{aligned} \mathbf{x}(k+1) &= \Phi\mathbf{x}(k) + \Gamma\left[\mathbf{u}(k) + \mathbf{w}(k)\right] \\ \mathbf{x}(0) &= \mathbf{X}(0) - \overline{\mathbf{X}} \end{aligned}$$

OR by solving nonlinear ODEs (1-4) with the given initial condition. In the later case, the input entering the plant is determined as follows

$$\mathbf{U}(k) = \overline{\mathbf{U}} + \mathbf{u}(k) + \mathbf{w}(k) \quad \text{for } kT \leq t < (k+1)T$$

where T represents sampling interval. Use T = 5 units.

- **State Estimation** Implement Kalman predictor by assuming

$$\begin{aligned} \widehat{\mathbf{x}}(0|-1) &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \\ P(0|-1) &= 10\mathbf{I} \end{aligned}$$

and estimate sequence $\{\widehat{\mathbf{x}}(k|k-1) : k = 1, ...100\}$.

– Graphically compare true $\mathbf{x}_i(k)$ v/s time and estimated states $\widehat{\mathbf{x}}_i(k|k-1)$ v/s time for i = 1,2 (i.e. the unmeasured states).

– Plot estimation error $\boldsymbol{\varepsilon}_i(k|k-1)$ v/s time for i = 1,2.

Kalman Predictor

$$\mathbf{L}_p(k) = \mathbf{\Phi}\mathbf{P}(k|k-1)\mathbf{C}^T\left[\mathbf{C}\mathbf{P}(k|k-1)\mathbf{C}^T + \mathbf{R}\right]^{-1}$$

$$\begin{aligned} \mathbf{e}(k) &= \mathbf{y}(k) - \mathbf{C}\widehat{\mathbf{x}}(k|k-1) \\ \widehat{\mathbf{x}}(k+1|k) &= \mathbf{\Phi}\widehat{\mathbf{x}}(k|k-1) + \mathbf{\Gamma}_u\mathbf{u}(k) + \mathbf{L}_p\mathbf{e}(k) \end{aligned}$$

$$\mathbf{P}(k+1|k) = \mathbf{\Phi}\mathbf{P}(k|k-1)\mathbf{\Phi}^T + \Gamma_d\mathbf{Q}\mathbf{\Gamma}_d^T - \mathbf{L}(k)\mathbf{C}\mathbf{P}(k|k-1)\mathbf{\Phi}^T$$

Note: *Since the unmeasured disturbance is in the manipulated inputs,* $\Gamma_d = \Gamma$ *in this example.*