Advanced Process Control
Quiz
One hour Close Book Examination

- **Linear Perturbation Models**

  - *Continuous Time*

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax} + \mathbf{Bu} + \mathbf{Hd} \tag{1}$$
$$\mathbf{y}(t) = \mathbf{Cx}(t) \tag{2}$$

  - *Discrete Time*

$$\mathbf{x}(k+1) = \mathbf{\Phi x}(k) + \mathbf{\Gamma u}(k) \tag{3}$$
$$\mathbf{y}(k) = \mathbf{Cx}(k) \tag{4}$$

- **Discretization methods**

  - *Exact method*

$$\mathbf{\Phi} = \Psi \exp(\Lambda t)\Psi^{-1}$$
$$\Gamma = \int_0^T \exp(\mathbf{A}\tau)\mathbf{B}\ d\tau = (\mathbf{\Phi} - \mathbf{I})\mathbf{A}^{-1}\mathbf{B}$$

  - *Euler method*

$$\mathbf{\Phi} = \mathbf{I} + T\mathbf{A} \qquad \text{and} \qquad \Gamma = T\mathbf{B}$$

- Frobenious norm of a $(n \times m)$ matrix $\mathbf{A}$ with elements a_{ij} is given as

$$\|\mathbf{A}\|_F = \left[\sum_i \sum_j |a_{ij}|^2\right]^{1/2}$$

1. Dynamic model for a mechanical system (single link manipulator with flexible joints) is given by

$$I\frac{d^2\theta_1}{dt^2} + MgL\sin(\theta_1) + \kappa(\theta_1 - \theta_2) = 0$$
$$J\frac{d^2\theta_2}{dt^2} - \kappa(\theta_1 - \theta_2) = U$$

   where u represents manipulated torque input and (I, J, M, L, κ) represent model parameters.

(a) Defining state variables as

$$\mathbf{X}_1 = \theta_1, \mathbf{X}_2 = \frac{d\theta_1}{dt}, \mathbf{X}_3 = \theta_2, \mathbf{X}_4 = \frac{d\theta_4}{dt}$$

and measured outputs as

$$\mathbf{Y}_1 = \mathbf{X}_1, \mathbf{Y}_2 = \mathbf{X}_3$$

rearrange the dynamic model in the standard form

$$\begin{aligned} \frac{d\mathbf{X}_i}{dt} &= \mathbf{f}_i(\mathbf{X}, U) \quad \text{for} \quad i = 1, 2, 3, 4 \\ \mathbf{Y} &= \mathbf{CX} \end{aligned}$$

where $\mathbf{f}_i(\mathbf{X}, U)$ represents i'th elements of function vector $\mathbf{F}(\mathbf{X}, U)$ and $\mathbf{C}$ represents observation matrix that relates states to the measured outputs. (4 points)

Hint: *Note that $d\mathbf{X}_1/dt = \mathbf{X}_2$ and a similar relationship exists between $\mathbf{X}_3$* and $\mathbf{X}_4$.

(b) Find the Jacobian matrices

$$\mathbf{A} = \left[\frac{\partial \mathbf{F}}{\partial \mathbf{X}}\right] \quad \text{and} \quad \mathbf{B} = \left[\frac{\partial \mathbf{F}}{\partial U}\right]$$

appearing in the linear perturbation model (1-2). Express elements of these matrices in terms of the steady state vector $\overline{\mathbf{X}} = \begin{bmatrix} \overline{\mathbf{X}}_1 & \overline{\mathbf{X}}_2 & \overline{\mathbf{X}}_3 & \overline{\mathbf{X}}_4 \end{bmatrix}^T$ and steady state input $\overline{U}$ (5)

2. Consider a **continuos time** linear perturbation model with

$$\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

(a) Diagonalize matrix $\mathbf{A}$ as $\mathbf{A} = \Psi\Lambda\Psi^{-1}$ using eigen values and eigen vectors of $\mathbf{A}$. Comment upon the asymptotic stability behavior of the unforced system using eigen values of matrix A. (4 points)

(b) It is desired to convert the given continuous time model to a discrete time model of the form (3-4) using the exact method. Find matrices Φ, Γ for the discrete time linear model with $T = 1$ using exact method and report eigen values of Φ. (4 points)

(c) Suppose discretization of the continuous time system is carried out using the Euler's method i.e. $[\Phi]_{Euler} = \mathbf{I} + T\mathbf{A}$. Find the error in discretization caused by Euler's approximation

$$\Delta\Phi = [\Phi]_{Exact} - [\Phi]_{Euler}$$

and find the Frobenious norm of error matrix $\Delta\Phi$. (3 points)