

Advanced Process Control

(Time: 3 hours)

End Semester Examination (40 Marks)

Instructions. *Closed Book and Closed Notes* examination.

Solution Key

1. **Pole placement controller and Kalman predictor design:** Consider the following difference equation

$$\mathbf{x}(k+1) = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} -2 \\ 2 \end{bmatrix} u(k) + \begin{bmatrix} 2 \\ 2 \end{bmatrix} w(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(k) + v(k)$$

$$w(k) \sim N(0, 1/4) \quad \text{and} \quad v(k) \sim N(0, 1/4)$$

- (a) It is desired to develop a state feedback control law of the form

$$u(k) = -G\mathbf{x}(k)$$

Find matrix G such that the poles of $(\Phi - \Gamma G)$ are placed at $q = 0.25 \pm 0.25j$.
(6 marks)

Soln.: Approach 1: Let $G = \begin{bmatrix} a & b \end{bmatrix}$. Then, the closed loop matrix is

$$\begin{aligned} & (\Phi - \Gamma G) \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} \\ &= \begin{bmatrix} 1/2 + 2a & 1/2 + 2b \\ -1/2 - 2a & 1/2 - 2b \end{bmatrix} \end{aligned}$$

and the corresponding characteristic equation is

$$\det[\lambda \mathbf{I} - (\Phi - \Gamma G)] = 0$$

$$(\lambda - 1/2 - 2a)(\lambda - 1/2 + 2b) + (1/2 + 2b)(1/2 + 2a) = 0$$

$$\lambda^2 + [(-1/2 - 2a) + (-1/2 + 2b)]\lambda + C = 0$$

$$\lambda^2 + [-1 - 2a + 2b]\lambda + C = 0$$

$$C = -(1/2 + 2a)(-1/2 + 2b) + (1/2 + 2b)(1/2 + 2a)$$

$$= -(-1/4 + b - a + 4ab) + (1/4 + a + b + 4ab)$$

$$= 1/2 + 2a$$

$$\lambda^2 + [-1 - 2a + 2b]\lambda + 1/2 + 2a = 0$$

The desired characteristic polynomial is

$$(\lambda - (1/4) - (1/4)j)(\lambda - (1/4) + (1/4)j) = 0$$

$$(\lambda - (1/4))^2 + (1/16) = 0$$

$$\lambda^2 - (\lambda/2) + 1/8 = 0$$

Equating coefficients of the closed loop characteristic polynomial with the desired characteristic polynomial, we have

$$\begin{aligned} 1/2 + 2a &= 1/8 \Rightarrow a = -3/16 \\ [-1 - 2a + 2b] &= -1/2 \\ 2(b - a) &= 1/2 \Rightarrow b = 1/16 \end{aligned}$$

or

$$G = \begin{bmatrix} -3/16 & 1/16 \end{bmatrix} = \begin{bmatrix} -0.1875 & 0.0625 \end{bmatrix}$$

Approach 2: Transfer function for the given system is

$$y(k) = \frac{2q}{q^2 - q + 0.5}u(k)$$

and the controllable canonical realization becomes

$$\begin{aligned} \tilde{\mathbf{x}}(k+1) &= \begin{bmatrix} 1 & -1/2 \\ 1 & 0 \end{bmatrix} \tilde{\mathbf{x}}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 2 & 0 \end{bmatrix} \tilde{\mathbf{x}}(k) \end{aligned}$$

The observer designed in the transformed domain

$$\tilde{G} = \begin{bmatrix} -1/2 & 3/8 \end{bmatrix}$$

and the transformation matrix to recover the G is

$$\mathbf{T} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

- (b) For the dynamic system described above, **set up equations** for designing the steady state Kalman predictor of the form

$$\hat{\mathbf{x}}(k+1) = \Phi \hat{\mathbf{x}}(k) + \Gamma \mathbf{u}(k) + \mathbf{L} [y(k) - \mathbf{C} \hat{\mathbf{x}}(k)]$$

where \mathbf{L} represents steady state Kalman gain. (6 marks).

Note: Algebraic Riccati Equations are as follows

$$\begin{aligned} \mathbf{P} &= \Phi \mathbf{P} \Phi^T + \mathbf{Q} - \mathbf{L} \mathbf{C} \mathbf{P} \Phi^T \\ \mathbf{L} &= \Phi \mathbf{P} \mathbf{C}^T (\mathbf{R} + \mathbf{C} \mathbf{P} \mathbf{C}^T)^{-1} \end{aligned}$$

where matrix \mathbf{P} is of the form

$$\mathbf{P} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

You are NOT expected to solve the resulting equations. Only state the equations in terms of unknowns (a,b,c).

Soln.: Let

$$\mathbf{P} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{R} + \mathbf{CPC}^T &= (1/4) + [0 \ 1] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= (1/4) + c = (1/4) + c = \frac{4c+1}{4} \end{aligned}$$

$$\begin{aligned} \Phi\mathbf{P}\mathbf{C}^T &= \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} \\ &= \begin{bmatrix} (b+c)/2 \\ (c-b)/2 \end{bmatrix} \end{aligned}$$

$$L = \frac{2}{(4c+1)} \begin{bmatrix} (b+c) \\ (c-b) \end{bmatrix}$$

$$\begin{aligned} \mathbf{Q} &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} (1/4) [2 \ 2] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ \Phi\mathbf{P}\Phi^T &= \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} (1/2)(a+b) & (1/2)(b+c) \\ 1/2(b-a) & 1/2(c-b) \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} (1/4)(a+2b+c) & (1/4)(c-a) \\ 1/4(c-a) & 1/4(c-2b+a) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{LCP}\Phi^T &= \frac{2}{(4c+1)} \begin{bmatrix} (b+c) \\ (c-b) \end{bmatrix} [0 \ 1] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \\ &= \frac{2}{(4c+1)} \begin{bmatrix} (b+c) \\ (c-b) \end{bmatrix} [b \ c] \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \\ &= \frac{2}{(4c+1)} \begin{bmatrix} (b+c) \\ (c-b) \end{bmatrix} [(1/2)(b+c) \ (1/2)(c-b)] \\ &= \frac{2}{(4c+1)} \begin{bmatrix} (1/2)(b+c)^2 & (1/2)(c^2-b^2) \\ (1/2)(c^2-b^2) & (1/2)(c-b)^2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} a & b \\ b & c \end{bmatrix} &= \begin{bmatrix} (1/4)(a+2b+c) & (1/4)(c-a) \\ 1/4(c-a) & 1/4(c-2b+a) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &\quad - \frac{2}{(4c+1)} \begin{bmatrix} (1/2)(b+c)^2 & (1/2)(c^2-b^2) \\ (1/2)(c^2-b^2) & (1/2)(c-b)^2 \end{bmatrix} \end{aligned}$$

We arrive at the following three equations in three unknowns

$$\begin{aligned} a &= (1/4)(a + 2b + c) + 1 - \frac{(b + c)^2}{(4c + 1)} \\ b &= (1/4)(c - a) + 1 - \frac{(c^2 - b^2)}{(4c + 1)} \\ c &= 1/4(c - 2b + a) + 1 - \frac{(b - c)^2}{(4c + 1)} \end{aligned}$$

which have to be solved simultaneously.

2. Stability, State Estimation and State Realization

- (a) Consider the following difference equation representing dynamic behavior of a satellite.

$$\begin{aligned} \mathbf{x}(k+1) &= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} (T^2)/2 \\ T \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k) \end{aligned}$$

where T represents the sampling interval. Is this system observable and reachable for any choice of the sampling time T ? (4 marks)

Soln.: Observability matrix for the system is

$$W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & T \end{bmatrix}$$

$$\text{Det}(W_o) = T \neq 0 \text{ for any } T > 0$$

$$\text{Rank}(W_o) = 2 \text{ for any } T > 0$$

and the system is observable for any choice of sampling time $T > 0$.

Reachability matrix for the system is

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix} = \begin{bmatrix} T^2/2 & 3T^2/2 \\ T & T \end{bmatrix}$$

$$\text{Det}(W_c) = -T^3 \neq 0 \text{ for any } T > 0$$

$$\text{Rank}(W_c) = 2 \text{ for any } T > 0$$

and the system is reachable for any choice of sampling time $T > 0$.

- (b) Consider a discrete time system

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.5 & 1 \\ -0.5 & 0.5 \end{bmatrix} \mathbf{x}(k)$$

Determine whether the following function below qualifies to be a Lyapunov function for this system. (5 marks)

$$V[\mathbf{x}(k)] = [x_2(k)]^2 + [x_1(k) + 2x_2(k)]^2$$

Soln.: Approach 1: Since $V[\mathbf{x}(k)]$ is expressed as sum of two squares,

$$V[\mathbf{x}(k)] > 0 \text{ for any } \mathbf{x}(k) \neq \bar{\mathbf{0}}$$

can be expressed as follows

$$\begin{aligned} V[\mathbf{x}(k)] &= [x_1(k)]^2 + 4x_1(k)x_2(k) + 5[x_2(k)]^2 \\ &= \mathbf{x}(k)^T \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \mathbf{x}(k) \end{aligned}$$

$$\Delta V(\mathbf{x}) = \mathbf{x}(k)^T [\Phi^T \mathbf{P} \Phi - \mathbf{P}] \mathbf{x}(k)$$

$$\begin{aligned} \Phi^T \mathbf{P} \Phi - \mathbf{P} &= \begin{bmatrix} 1/2 & -5/4 \\ -5/4 & 17/4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -1/2 & -13/4 \\ -13/4 & -3/4 \end{bmatrix} \end{aligned}$$

Checking definiteness of matrix $\Phi^T \mathbf{P} \Phi - \mathbf{P}$:

Method 1: Find eigenvalues of $(\Phi^T \mathbf{P} \Phi - \mathbf{P})$

$$\begin{aligned} \det[\lambda \mathbf{I} - (\Phi^T \mathbf{P} \Phi - \mathbf{P})] &= 0 \\ \lambda^2 + \frac{5}{4}\lambda - \frac{163}{16} &= 0 \end{aligned}$$

Roots of this equation are

$$\lambda_1 = -3.8774 \text{ and } \lambda_2 = 2.6274$$

Thus, $(\Phi^T \mathbf{P} \Phi - \mathbf{P})$ is an indefinite matrix and $V[\mathbf{x}(k)]$ does not qualify to be a Lyapunov function.

Method 2: Check for positive definiteness of $-(\Phi^T \mathbf{P} \Phi - \mathbf{P})$

$$-(\Phi^T \mathbf{P} \Phi - \mathbf{P}) = \begin{bmatrix} 1/2 & 13/4 \\ 13/4 & 3/4 \end{bmatrix}$$

Checking determinant of the minors

$$\begin{aligned} 1/2 &> 0 \\ (1/2)(3/4) - (169/16) &< 0 \end{aligned}$$

Thus, $-(\Phi^T \mathbf{P} \Phi - \mathbf{P})$ is indefinite and $V[\mathbf{x}(k)]$ does not qualify to be a Lyapunov function.

Approach 2: Alternatively, $V[\mathbf{x}(k)]$ can be expressed as follows

$$\begin{aligned} V[\mathbf{x}(k)] &= [x_1(k)]^2 + 4x_1(k)x_2(k) + 5[x_2(k)]^2 \\ &= \mathbf{x}(k)^T \begin{bmatrix} 1 & 4 \\ 0 & 5 \end{bmatrix} \mathbf{x}(k) \end{aligned}$$

and the procedure outlined in Approach 1 can be followed. The conclusions reached regarding suitability of $V[\mathbf{x}(k)]$ as Lyapunov function through this choice of representing \mathbf{P} matrix are identical to that of Approach 1.

(c) For the following model identified from input-output data

$$y(k) = \frac{q-1}{(q-0.5)(q-0.4)}u_1(k) + \frac{2q+1}{(q-0.4)}u_2(k) + \frac{(q+0.6)}{(q-0.5)}e(k)$$

where $\{e(k)\}$ is a zero mean white noise sequence with variance 0., derive state realization

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k) + \mathbf{L}e(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}u(k) + e(k) \end{aligned}$$

in the **observable canonical form**. (7 marks)

Soln.: Given transfer function can be expressed as follows

$$\begin{aligned} y(k) &= \frac{q-1}{(q-0.5)(q-0.4)}u_1(k) + \frac{(2q+1)(q-0.5)}{(q-0.4)(q-0.5)}u_2(k) \\ &\quad + \frac{(q+0.6)(q-0.4)}{(q-0.5)(q-0.4)}e(k) \end{aligned}$$

$$A(q) = (q-0.5)(q-0.4) = q^2 - 0.9q + 0.2$$

$$(2q+1)(q-0.5) = 2q^2 - 0.5$$

$$(q+0.6)(q-0.4) = q^2 + 0.2q - 0.24$$

$$\begin{aligned} y(k) &= \frac{q-1}{A(q)}u_1(k) + \frac{2q^2-0.5}{A(q)}u_2(k) \\ &\quad + \frac{q^2+0.2q-0.24}{A(q)}e(k) \end{aligned}$$

Since degree of denominator and numerator polynomials are identical for the transfer function w.r.t. $u_2(k)$, it can be re-written as follows

$$\begin{aligned} \frac{2q^2-0.5}{A(q)} &= \frac{2q^2-0.5}{q^2-0.9q+0.2} - 2 + 2 \\ &= \frac{0.18q-0.9}{q^2-0.9q+0.2} + 2 \end{aligned}$$

Similarly,

$$\begin{aligned}\frac{q^2 + 0.2q - 0.24}{A(q)} &= \frac{q^2 + 0.2q - 0.24}{q^2 - 0.9q + 0.2} - 1 + 1 \\ &= \frac{1.1q - 0.44}{q^2 - 0.9q + 0.2} + 1\end{aligned}$$

Thus, the given model can be rewritten as follows

$$\begin{aligned}\tilde{y}(k) &= \frac{q-1}{A(q)}u_1(k) + \frac{0.18q-0.9}{A(q)}u_2(k) \\ &\quad + \frac{1.1q-0.44}{A(q)}e(k) \\ y(k) &= \tilde{y}(k) + 2u_2(k) + e(k)\end{aligned}$$

Observable canonical realization for $\tilde{y}(k)$ can be written by observing coefficients of transfer functions

$$\begin{aligned}\mathbf{x}(k+1) &= \begin{bmatrix} 0.9 & 1 \\ -0.2 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 & 0.18 \\ -1 & -0.9 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} + \begin{bmatrix} 1.1 \\ -0.44 \end{bmatrix} e(k) \\ \tilde{y}(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} + e(k)\end{aligned}$$

3. Predictive Control and Model Matching Control Design

- (a) It is desired to develop a conventional MPC type predictive controller using a model of form

$$\begin{aligned}\mathbf{x}(k+1) &= \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 2 \\ 4 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{x}(k) + e(k)\end{aligned}$$

and open loop observer

$$\begin{aligned}\hat{\mathbf{x}}(k) &= \Phi \hat{\mathbf{x}}(k-1) + \Gamma u(k-1) \\ e(k) &= y(k) - C \hat{\mathbf{x}}(k)\end{aligned}$$

using two step ahead predictions and control horizon equal to two.

- i. At sampling instant k , derive two step ahead future predictions using the model with the initial condition as $\hat{\mathbf{x}}(k)$ and arrange the predictions in following matrix equation

$$\mathbf{Y}(k) = \begin{bmatrix} \hat{\mathbf{y}}(k+1|k) \\ \hat{\mathbf{y}}(k+2|k) \end{bmatrix} = \mathbf{S} \hat{\mathbf{x}}(k) + \mathbf{G} \mathbf{U}(k) + \mathbf{L} e(k)$$

where matrices \mathbf{S} , \mathbf{G} and \mathbf{L} are obtained using given model matrices (Φ, Γ, C) and

$$\mathbf{U}(k) = \begin{bmatrix} u(k|k) \\ u(k+1|k) \end{bmatrix}$$

(3 marks)

Soln.: In the conventional MPC formulation, to carry out predictions given future inputs

$$\{\mathbf{u}(k|k), \mathbf{u}(k+1|k)\}$$

, the model equations are used as follows

$$\begin{aligned}\hat{\mathbf{x}}(k+1|k) &= \Phi\hat{\mathbf{x}}(k) + \Gamma\mathbf{u}(k|k) \\ \hat{\mathbf{x}}(k+2|k) &= \Phi\hat{\mathbf{x}}(k+1|k) + \Gamma\mathbf{u}(k+1|k) \\ &= \Phi^2\hat{\mathbf{x}}(k) + \Phi\Gamma\mathbf{u}(k|k) + \Gamma\mathbf{u}(k+1|k)\end{aligned}$$

Using model residue $\mathbf{e}(k)$ for compensating model plant mismatch, output predictions become

$$\begin{aligned}\hat{\mathbf{y}}(k+1|k) &= C\Phi\hat{\mathbf{x}}(k) + C\Gamma\mathbf{u}(k|k) + \mathbf{e}(k) \\ \hat{\mathbf{y}}(k+2|k) &= C\Phi^2\hat{\mathbf{x}}(k) + C\Phi\Gamma\mathbf{u}(k|k) + C\Gamma\mathbf{u}(k+1|k) + \mathbf{e}(k)\end{aligned}$$

$$\begin{aligned}\begin{bmatrix} \hat{\mathbf{y}}(k+1|k) \\ \hat{\mathbf{y}}(k+2|k) \end{bmatrix} &= \begin{bmatrix} C\Phi \\ C\Phi^2 \end{bmatrix} \hat{\mathbf{x}}(k) + \begin{bmatrix} \mathbf{I} \\ \mathbf{I} \end{bmatrix} \mathbf{e}(k) \\ &\quad + \begin{bmatrix} C\Gamma & 0 \\ C\Phi\Gamma & C\Gamma \end{bmatrix} \begin{bmatrix} \mathbf{u}(k|k) \\ \mathbf{u}(k+1|k) \end{bmatrix}\end{aligned}$$

ii. Let

$$\mathbf{R}(k) = \begin{bmatrix} \mathbf{r}(k+1) \\ \mathbf{r}(k+2) \end{bmatrix}$$

define vector of future setpoints. Then, a control law that minimizes two norm of the future prediction error

$$\mathbf{E}(k) = \mathbf{R}(k) - \mathbf{Y}(k)$$

is given by setting future error $\mathbf{E}(k) = \bar{\mathbf{0}}$. Thus, find vector $\mathbf{U}(k)$ that will meet constraint $\mathbf{R}(k) = \mathbf{Y}(k)$ at each sampling instant. (3 marks).

Soln.: Substituting the values

$$\begin{aligned}\begin{bmatrix} \hat{\mathbf{y}}(k+1|k) \\ \hat{\mathbf{y}}(k+2|k) \end{bmatrix} &= \begin{bmatrix} \mathbf{0.5} & -0.5 \\ 0.25 & -0.25 \end{bmatrix} \hat{\mathbf{x}}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{e}(k) \\ &\quad + \begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix} \mathbf{U}(k)\end{aligned}$$

Using constraint

$$\mathbf{R}(k) = \mathbf{Y}(k)$$

it follows that

$$\begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix} \mathbf{U}(k) = \mathbf{R}(k) - \begin{bmatrix} \mathbf{0.5} & -0.5 \\ 0.25 & -0.25 \end{bmatrix} \hat{\mathbf{x}}(k) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{e}(k)$$

$$\mathbf{U}(k) = \begin{bmatrix} -\mathbf{0.5} & 0 \\ 0.25 & -0.5 \end{bmatrix} \mathbf{R}(k) - \begin{bmatrix} -\mathbf{0.25} & 0.25 \\ 0 & 0 \end{bmatrix} \hat{\mathbf{x}}(k) - \begin{bmatrix} -0.5 \\ -0.25 \end{bmatrix} \mathbf{e}(k)$$

(b) Consider process governed by

$$\mathbf{x}(k+1) = \Phi \mathbf{x}(k) + \Gamma \mathbf{u}(k) \quad (1)$$

$$\mathbf{y}(k) = \mathbf{C} \mathbf{x}(k) \quad (2)$$

Further assume that number of manipulated inputs equals the number of controlled outputs. It is desired to arrive at a state feedback control law such that the output dynamics is governed by

$$\mathbf{y}(k+1) = \mathbf{A} \mathbf{y}(k) + (\mathbf{I} - \mathbf{A}) \mathbf{r}(k) \quad (3)$$

where $\mathbf{r}(k)$ represents the setpoint.

- i. Find $\mathbf{u}(k)$ as a function of $\mathbf{x}(k)$ such that the dynamics of $\mathbf{y}(k)$ generated by combining equations (1)-(2) exactly matches dynamics of $\mathbf{y}(k)$ given by (3). Does it yield a state feedback control law? (4 marks)

Soln.: *Combining model equations*

$$\begin{aligned} \mathbf{C} \mathbf{x}(k+1) &= \mathbf{y}(k+1) = \mathbf{C} \Phi \mathbf{x}(k) + \mathbf{C} \Gamma \mathbf{u}(k) \\ &= \mathbf{A} \mathbf{y}(k) + (\mathbf{I} - \mathbf{A}) \mathbf{r}(k) \\ &= \mathbf{A} \mathbf{C} \mathbf{x}(k) + (\mathbf{I} - \mathbf{A}) \mathbf{r}(k) \end{aligned}$$

$$\mathbf{C} \Gamma \mathbf{u}(k) = [\mathbf{A} \mathbf{C} - \mathbf{C} \Phi] \mathbf{x}(k) + (\mathbf{I} - \mathbf{A}) \mathbf{r}(k)$$

Since number of inputs are equal to the number of outputs, if matrix $\mathbf{C} \Gamma$ is invertible, then it follows that

$$\mathbf{u}(k) = [\mathbf{C} \Gamma]^{-1} \{[\mathbf{A} \mathbf{C} - \mathbf{C} \Phi] \mathbf{x}(k) + (\mathbf{I} - \mathbf{A}) \mathbf{r}(k)\}$$

Note: *In general, matrices \mathbf{C} and Γ are NOT square matrices and hence not invertible. Thus, any solution that involves inverses of these matrices is not acceptable as no mention was made in the question regarding the dimensions of these matrices.*

- ii. Is there any condition that needs to be satisfied by some of the model matrices for such control law to exist? (2 marks)

• **Soln.**

$$\begin{aligned} \text{Rank}(\mathbf{C} \Gamma) &= \text{No. of Manipulated Inputs} \\ &= \text{No. of Controlled Outputs} \end{aligned}$$

$$[\mathbf{A} \mathbf{C} - \mathbf{C} \Phi] \neq [\mathbf{0}] \quad (\text{i.e. Null Matrix})$$